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Algebras satisfying triality and S_4 symmetry

In memory of Professor Susumu Okubo (1930-2015)

by

Noriaki Kamiya
and Susumu Okubo

Eulogy

One of the authors (N.Kamiya) had been met and
discussed with Prof.Susumu Okubo a lot of times.
For study in nonaasociative algebra's subject and
private talk in Japanese with me,
He was always honest man.
Hereby I would like to give an eulogy in this book.
Also we remember him when working octonion or
triple systems in mathematical physics.

Hereby we would like to describe the last words of the end in his life(Prof. Susumu Okubo).

Jisei no ku

To be or not to be ? The quantum dream of the Schrodinger Cat.
Farewell! Farewell-forever. Departure time now to the black hole.
Never to return, farewell, sayonara.

Algebras satisfying Triality and S_4-symmetry

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Abstract

We give a review book of recent works, which contains our new results for nonassociative algebras, especially Lie algebras satisfying the triality relation. They are also intimately related to S_4 (symmetric group of 4-objects) symmetry of the Lie algebras.

Keywords

Structurable algebras, Freudenthal-Kantor triple systems, Lie algebras and superalgebras, Triality, S_4-symmetry.

Contents

This monoglaph is one in the end of his life (Prof. S.Okubo).

1. Symmetric Triality Algebras

This book is intented to be a brief review of recent works with new results related to triality relations as well as to A_4 or S_4-symmetry (alternative or symmetric group of 4-objects) satisfied by some algebras, especially, Lie algebras. And main purpose is to exhebit constractions of Lie algebras or superalgebras without utilizing properties of Cartan matrices and root systems.

More precisely, we introduce the notion of triality algebras and describe constructions of Lie algebras or superalgebras from them.

First let A be an algebra over a field F with bi-linear product denoted by juxtaposition xy. Suppose that some $t_j \in$ End A for $j = 0, 1, 2$ satisfy the symmetric triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y) \tag{1.1}$$

for any $x, y \in A$, where indices j are defined modulo 3, i.e.

$$t_{j\pm 3} = t_j.$$

We then call the triple $t = (t_1, t_2, t_3) \in (\text{End } A)^3$ be a symmetric Lie-related triple ([O.05]) and set

$$s \circ Lrt(A) = \{t = (t_1, t_2, t_3) | t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y)\}. \tag{1.2}$$

For any two $t, t' \in s \circ Lrt(A)$, $T_{j,k} \in$ End A defined by

$$T_{j,k} = [t_j, t_k'] := t_j t_k' - t_k' t_j, \ (j, k = 0, 1, 2) \tag{1.3a}$$

then satisfy

$$T_{j,k}(xy) = (T_{j+1,k+1}x)y + x(T_{j+2,k+2}y). \tag{1.3b}$$

Especially $s \circ Lrt(A)$ is a Lie algebra with respect to the component-wise commutation relation. Moreover, it is endowed with a natural order 3 automorphysm θ given by

$$\theta(t_0, t_1, t_2) = (t_2, t_0, t_1).$$

It is a generalization of the derivation Lie algebra

$$Der(A) = \{d | d(xy) = (dx)y + x(dy), \ \forall x, y \in A, \ d \in \text{End}A\}.$$

For any constants $\lambda_j \in F$ satisfying the condition $\lambda_{j\pm 3} = \lambda_j$, then $t' = (t'_0, t'_1, t'_2)$ defined by

$$t'_j = \sum_{k=0}^{2} \lambda_{j-k} t_k$$

belongs also to $t' \in s \circ Lrt(A)$, when we note

$$t' = (t'_0, t'_1, t'_2) = \sum_{j=0}^{2} \lambda_j \theta^j (t_0, t_1, t_2).$$

Especially, for the choice of $\lambda_0 = \lambda_1 = \lambda_2 = 1$,

$$d = t_0 + t_1 + t_2 \tag{1.4}$$

is a derivation of A, i.e. $d \in \text{Der}\,(A)$.

Suppose now that A is also involutive with the involution map $x \to \overline{x}$, satisfying

$$\overline{\overline{x}} = x, \;\; \bar{x}\bar{y} = \overline{yx}. \tag{1.5}$$

For any $Q \in \text{ End } A$, we introduce $\overline{Q} \in \text{ End } A$ by

$$\overline{Qx} = \overline{Q}\bar{x}. \tag{1.6}$$

We then note that for any two $Q_1, Q_2 \in \text{ End } A$, we have

$$\overline{Q_1 Q_2} = \overline{Q}_1 \overline{Q}_2. \tag{1.7}$$

Taking the involution of Eq.(1.1) and then letting $x \leftrightarrow \overline{y}$. it gives

$$\bar{t}_j(xy) = (\bar{t}_{j+2}x)y + x(\bar{t}_{j+1}y) \tag{1.8}$$

so that

$$(\bar{t}_0, \bar{t}_2, \bar{t}_1) \in s \circ Lrt(A). \tag{1.9}$$

If we introduce $\sigma \in \text{ End}(s \circ Lrt(A))$ by $\sigma(t_0, t_1, t_2) = (\bar{t}_0, \bar{t}_2, \bar{t}_1)$, then σ and θ generate automorphism group S_3 of $s \circ Lrt(A)$, since we have $\theta\sigma\theta = \sigma$, and $\theta^3 = \sigma^2 = id$.

We next introduce the second bi-linear product in the vector space of A by

$$x \star y := \overline{xy} = \bar{y}\,\bar{x}. \tag{1.10}$$

Then the resulting algebra A^* which we call the conjugate algebra of A is also involutive, i.e.,

$$\overline{x \star y} = \bar{y} \star \bar{x} (= xy) \tag{1.11}$$

where Eq.(1.8) is rewritten as the Lie-related triple relation ([A-F.93])).

$$\bar{t}_j(x \star y) = (t_{j+1}x) \star y + x \star (t_{j+2}y). \tag{1.12}$$

A reason for introducing A^* is due to the following consideration.

If A is a unital algebra over the field F of characteristic not 2, then it is easy to show that we have $t_0 = t_1 = t_2$ for $s \circ Lrt(A)$ and hence $s \circ Lrt(A) \cong \mathrm{Der}(A)$, since for the unit element e of A, it holds $t_j(xe) = t_{j+1}(x)e + xt_{j+2}(e)$ and $t_j(e) = 0,\ j = 0, 1, 2$.

However, A^* could be unital, satisfying $e \star x = x \star e = x$ without leading to $t_0 = t_1 = t_2$. Then the relation is immediately translated into A to yield

$$ex = xe = \bar{x}. \tag{1.13}$$

We call $e \in A$ satisfying Eq.(1.13) be the para-unit of A.

Introducing multiplication operators in A and A^* by

$$L(x)y = xy, \quad R(x)y = yx, \tag{1.14a}$$

$$l(x)y = x \star y, \quad r(x)y = y \star x, \tag{1.14b}$$

they satisfy

$$L(x)R(y) = r(\bar{x})r(y) \tag{1.15a}$$

$$R(x)L(y) = l(\bar{x})l(y). \tag{1.15b}$$

We note that an algebra A could have more than one involution. Moreover, it is often easier to deal with A rather than A^* and we will discuss mostly relations involving A in this section, although they can be readily translated into those of A^*.

Lemma 1.1

For any $(t_0, t_1, t_2) \in s \circ Lrt(A)$, we have

$$[t_j, L(x)R(y)] = L(x)R(t_{j+1}y) + L(t_{j+1}x)R(y) \tag{1.16a}$$

$$[t_j, R(x)L(y)] = R(x)L(t_{j+2}y) + R(t_{j+2}x)L(y). \tag{1.16b}$$

Proof

We can rewrite Eq.(1.1) as

$$t_j L(x) = L(x)t_{j+2} + L(t_{j+1}x), \tag{1.17a}$$

$$t_j R(y) = R(y)t_{j+1} + R(t_{j+2}y). \tag{1.17b}$$

Multiplying $R(y)$ to Eq.(1.17a) from the right and $L(x)$ to Eq.(1.17b) from the left, we obtain

$$\begin{aligned} t_j L(x)R(y) &= L(x)t_{j+2}R(y) + L(t_{j+1}x)R(y), \\ L(x)t_j R(y) &= L(x)R(y)t_{j+1} + L(x)R(t_{j+2}y). \end{aligned}$$

Letting $j \to j+2$ in the 2nd relation and adding it to the first one this yield Eq.(1.16a). Similarly from Eqs.(1.17), we find

$$\begin{aligned} R(y)t_j L(x) &= R(y)L(x)t_{j+2} + R(y)L(t_{j+1}x), \\ t_j R(y)L(x) &= R(y)t_{j+1}L(x) + R(t_{j+2}y)L(x). \end{aligned}$$

Letting $j \to j+1$ in the first relation and adding it to the second one, we obtain Eq.(1.16b).□

Def.1.2

Let A be an algebra which possess $d_j(x, y) \in \text{End}\, A$ for $j = 0, 1, 2$ and for $x, y \in A$, satisfying
(1)

$$d_j(y, x) = -d_j(x, y) \tag{1.18a}$$

(2)

$$d_1(x, y) = R(y)L(x) - R(x)L(y) \tag{1.18b}$$

$$d_2(x, y) = L(y)R(x) - L(x)R(y) \tag{1.18c}$$

(3)

$$(d_0(x, y), d_1(x, y), d_2(x, y)) \in s \circ Lrt(A),$$

i.e, we have

$$d_j(x, y)(uv) = (d_{j+1}(x, y)u)v + u(d_{j+2}(x, y)v). \tag{1.19}$$

We call A then be a regular triality algebra. Note that a explicit form for $d_0(x, y)$ is *not* specified at all.

If A satisfies further

(4)

$$[d_j(u,v), d_k(x,y)] = d_k(d_{j-k}(u,v)x, y) + d_k(x, d_{j-k}(u,v)y), \tag{1.20}$$

and

(5)

$$d_0(x,y)z + d_0(y,z)x + d_0(z,x)y = 0 \tag{1.21}$$

in addition for any $j,k = 0,1,2$ and any $u,v,x,y,z \in A$, then A is called a pre-normal triality algebra.

The reason for introducing these definitions is due to the following considerations. To this end, we introduce

Condition (B)

We have $AA = A$.

Condition (C)

If some $b \in A$ satisfies either $bA = 0$ or $Ab = 0$, then $b = 0$.
We can now prove:

Proposition 1.3

Let A be a regular triality algebra satisfying the condition (C). Then, A is a pre-normal triality algebra. More generally, we obtain the followings:
If either condition (B) or (C) holds valid, we have

(1)

$$[t_j, d_k(x,y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y) \tag{1.22}$$

for any $t = (t_0, t_1, t_2) \in s \circ Lrt\ (A)$. Especially for a choice of $t_j = d_j(u,v)$, this implies the validity of Eq.(1.20).

(2) $d_0(x,y)$ is uniquely determined by Eqs.(1.18) and (1.19).

(3) If A is involutive in addition with the involution map $x \to \bar{x}$, we have

$$\overline{d_j(x,y)} = d_{3-j}(\bar{x}, \bar{y}). \tag{1.23}$$

(4) Finally, if we assume the condition (C), then $d_0(x,y)$ satisfies Eq.(1.21).

Proof

For a proof of this Proposition, we first set

$$D_{j,k} := [t_j, d_k(x,y)] - d_k(t_{j-k}x, y) - d_k(x, t_{j-k}y). \tag{1.24}$$

Then, Lemma 1.1 immediately gives $D_{j,1} = D_{j,2} = 0$ identically for any $j = 0, 1, 2$. Moreover Eq.(1.3b) for $t'_k = d_k(x,y)$ together with Eq.(1.19) leads to

$$D_{j,k}(uv) = (D_{j+1,k+1}u)v + u(D_{j+2,k+2}v). \tag{1.25}$$

Setting $k = 0$, we find $D_{j,0}(uv) = 0$ which gives $D_{j,0} = 0$ if the condition (B) holds. If we choose $k = 1$ or 2, then Eq.(1.25) implies

$$u(D_{j+2,0}v) = 0 = (D_{j+1,0}u)v$$

for any $j = 0, 1, 2$, and for any $u, v \in A$. Therefore, under the condition (C), this gives $D_{j,0} = 0$ again, proving the validity of Eq.(1.22).

Next, the uniqueness of $d_0(x,y)$ can be similarly proven as follows. Suppose that Eq.(1.18) and (1.19) allow the second solution for $d_0(x,y)$ which we write as $d'_0(x,y)$. Then,

$$(D_0, D_1, D_2) := (d_0(x,y) - d'_0(x,y), 0, 0) \in s \circ Lrt(A)$$

so that

$$D_j(uv) = (D_{j+1}u)v + u(D_{j+2}v).$$

Choosing $j = 0, 1$ or 2, and repeating the same reasoning, this gives $D_0 = 0$, i,e., $d'_0(x,y) = d_0(x,y)$.

If A is involutive, we have

$$\overline{L(x)} = R(\bar{x}),\ \overline{R(x)} = L(\bar{x}), \tag{1.26}$$

so that

$$\overline{d_1(x,y)} = d_2(\bar{x}, \bar{y}),\ \text{and}\ \overline{d_2(x,y)} = d_1(\bar{x}, \bar{y})$$

which satisfy Eq.(1.23) for $j = 1$ and 2. In order to show its validity for $j = 0$, we set

$$\tilde{D}_j := \overline{d_j(x,y)} - d_{3-j}(\overline{x}, \overline{y}) \quad (j = 0, 1, 2)$$

so that $\tilde{D}_1 = \tilde{D}_2 = 0$. Moreover, Eqs.(1.1) and (1.8) imply now that we have

$$\tilde{D}_j(xy) = (\tilde{D}_{j+2}x)y + x(\tilde{D}_{j+1}y).$$

Repeating again the same augument, we obtain $\tilde{D}_0 = 0$.

Finally, let us set

$$\Lambda(x) = \begin{pmatrix} 0, & R(x) \\ L(x), & 0 \end{pmatrix}, D_j(x,y) = \begin{pmatrix} d_j(x,y), & 0 \\ 0, & d_{j+1}(x,y) \end{pmatrix}. \tag{1.27}$$

Then, Eq.(1.19) with Eqs.(1.18) are equivalent to the validity of

$$[D_j(x,y), \Lambda(z)] = \Lambda(d_{j+2}(x,y)z) \tag{1.28a}$$

if we note Eqs.(1.17) for $t_j = d_j(u,v)$. Further, we see

$$[\Lambda(x), \Lambda(y)] = -D_1(x,y) \tag{1.28b}$$

so that

$$[\Lambda(z), [\Lambda(x), \Lambda(y)]] = \Lambda(d_0(x,y)z). \tag{1.28c}$$

If we set

$$w = d_0(x,y)z + d_0(y,z)x + d_0(z,x)y,$$

then the Jacobi identity among $\Lambda(x)'$s leads to $\Lambda(w) = 0$, or $R(w) = L(w) = 0$ so that we have $w = 0$ under the condition (C). This completes the proof of Proposition 1.3.□

We also note the Eq.(1.20) gives

$$[D_j(u,v), D_k(x,y)] = D_k(d_{j-k}(u,v)x, y) + D_k(x, d_{j-k}(u,v)y). \tag{1.29}$$

Therefore, $\Lambda(z)$ and $D_j(x,y)$ form a Lie algebra, although we will not go into details.

Let

$$g(A) = d_0(A,A) + d_1(A,A) + d_2(A,A). \tag{1.30}$$

Then, Eqs.(1.20) and (1.22) imply that $g(A)$ is a Lie algebra which is a ideal of the larger Lie algebra $s \circ Lrt(A)$. Moreover, if we set $j = k$ in Eq.(1.20), we obtain

$$[d_j(u,v), d_j(x,y)] = d_j(d_0(u,v)x, y) + d_j(x, d_0(u,v)y) \tag{1.31}$$

so that $d_j(A,A)$ for each $j = 0,1,2$ is also a Lie algebra which is a ideal of $g(A)$. Therefore if $g(A)$ is simple, and if $d_j(A,A) \neq 0$, then we must have

$$g(A) = d_0(A,A) = d_1(A,A) = d_2(A,A).$$

Corollary 1.4

Let A be a pre-normal triality algebra. Then, the triple product defined by

$$[xyz] := d_0(x,y)z$$

is a Lie triple product, i.e., it satisfies

$$\begin{array}{ll} (i) & [x,y,z] = -[y,x,z] \\ (ii) & [x,y,z]+[y,z,x]+[z,x,y]=0 \\ (iii) & [u,v,[x,y,z]] = [[u,v,x],y,z]+[x,[u,v,y],z]+[x,y,[u,v,z]]. \end{array}$$

Proof

First, (i) follows trivially since $d_0(y,x) = -d_0(x,y)$, while (ii) is a consequence of Eq.(1.21). Finally, (iii) is equivalent to the validity of Eq.(1.31) for $j=0$. □

We next set

$$D(x,y) := d_0(x,y)+d_1(x,y)+d_2(x,y). \tag{1.32}$$

Then, $D(x,y)$ is a derivation of A as d of Eq.(1.4). We further introduce $Q(x,y,z) \in$ End A by

$$Q(x,y,z) := d_0(x,yz)+d_1(z,xy)+d_2(y,zx). \tag{1.33}$$

Proposition 1.5

(1) If A is a regular triality algebra, then

$$(Q(x,y,z),Q(y,z,x),Q(z,x,y)) \in s \circ Lrt(A), i.e.,$$

$$Q(x,y,z)(uv) = (Q(y,z,x)u)v + u(Q(z,x,y)v). \tag{1.34}$$

Also,we have

$$Q(x,y,z)+Q(y,z,x)+Q(z,x,y) = D(x,yz)+D(y,zx)+D(z,xy). \tag{1.35}$$

(2) Moreover, if A is pre-normal triality algebra, then

$$Q(x,y,z)w = Q(w,y,z)x. \tag{1.36}$$

Further, if A is involutive with the validity of Eq.(1.23) in addition, it satisfies also

$$\overline{Q(x,y,z)} = Q(\bar{x},\bar{z},\bar{y}). \qquad (1.37)$$

Proof

By Eq.(1.19), we calculate

$$\begin{aligned} d_0(x,yz)(uv) &= (d_1(x,yz)u)v + u(d_2(x,yz)u) \\ d_1(z,xy)(uv) &= (d_2(z,xy)u)v + u(d_0(z,xy)v) \\ d_2(y,zx)(uv) &= (d_0(y,zx)u)v + u(d_1(y,zx)v). \end{aligned}$$

Adding all of these, we obtain Eq.(1.34). Similarly for Eq.(1.35). Since Eq.(1.23) gives $\overline{d_j(x,y)} = d_{3-j}(\overline{x},\overline{y})$, Eq.(1.37) follows immediately from Eq.(1.33).

Finally in order to prove Eq.(1.36), we calculate

$$\begin{aligned} &Q(x,y,z)w - Q(w,y,z)x = \{d_0(x,yz)w - d_0(w,yz)x\} \\ &+\{d_1(z,xy) + d_2(y,zx)\}w - \{d_1(z,wy) + d_2(y,zw)\}x, \qquad (1.38) \end{aligned}$$

and note

$$\begin{aligned} &d_0(x,yz)w - d_0(w,yz)x = -d_0(yz,x)w - d_0(w,yz)x \\ &= d_0(x,w)(yz) = \{d_1(x,w)y\}z + y\{d_2(x,w)z\} \end{aligned}$$

in view of Eqs.(1.21) and (1.19) for $j = 0$. Then, Eq.(1.38) becomes

$$\begin{aligned} &Q(x,y,z)w - Q(w,y,z)x \\ &= \{d_1(x,w)y\}z + y\{d_2(x,w)z\} + \{d_1(z,xy) + d_2(y,zx)\}w \\ &-\{d_1(z,wy) + d_2(y,zw)\}x \\ &= \{(R(w)L(x) - R(x)L(w))y\}z + y\{(L(w)R(x) - L(z)R(w))z\} \\ &+\{R(xy)L(z) - R(z)L(xy) + L(zx)R(y) - L(y)R(zx)\}w \\ &-\{R(wy)L(z) - R(z)L(wy) + L(zw)R(y) - L(y)R(zw)\}x \\ &= \{(xy)w\}z - \{(wy)x\}z + y\{w(zx)\} - y\{x(zw)\} \\ &+(zw)(xy) - \{(xy)w\}z + (zx)(wy) - y\{w(zx)\} \\ &-(zx)(wy) + \{(wy)x\}z - (zw)(xy) + y\{x(zw)\} = 0 \end{aligned}$$

identically. This completes the proof.□

We note that Eq.(1.34),(1.36) and (1.37) are consistent with the ansatz of $Q(x,y,z) = 0$, and we further define the following.

Def.1.6

We call a pre-normal triality algebra be a normal triality algebra if it satisfies $Q(x, y, z) = 0$ in addition. The conjugate algebra A^* of a normal triality algebra A satisfying Eq.(1.23) is called a normal Lie-related triality algebra (normal Lrt. algebra). More explicitly, it is defined by
(i)

$$d_1(x, y) = l(\bar{y})l(x) - l(\bar{x})l(y), \tag{1.39a}$$

$$d_2(x, y) = r(\bar{y})r(x) - r(\bar{x})r(y) \tag{1.39b}$$

(ii)

$$\overline{d_j(x, y)}(u \star v) = (d_{j+1}(x, y)u) \star v + u \star (d_{j+2}(x, y)v) \tag{1.39c}$$

(iii)

$$d_0(x, y)z + d_0(y, z)x + d_0(z, x)y = 0 \tag{1.39d}$$

(iv)

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y) \tag{1.39e}$$

(v)

$$Q(x, y, z) = d_0(x, \overline{y \star z}) + d_1(z, \overline{x \star y}) + d_2(y, \overline{z \star x}) = 0 \tag{1.39f}$$

(vi)

$$\overline{d_j(x, y)} = d_{3-j}(\overline{x}, \overline{y}). \tag{1.39g}$$

We note that Eqs.(1.39 a-f) are simple rewriting of the corresponding relations for the normal triality algebra A, when we note, for example, Eqs.(1.15) for Eq.s.(1.39). If A^* is unital with the unit element e, then both conditions (B) and (C) are automatically satisfied, because, by $xe = ex = \bar{x}$ for any x, and by $be = 0 \Rightarrow b = 0$. Then, we can omit Eqs.(1.39,d,e,and g) since they are consequence of other postulates by Proposition 1.3. Moreover, if we set $y = e$ or $z = e$ in Eq.(1.39f) or alternately if we set $u = e$ or $v = e$ in Eq.(1.39c), then $d_0(x, y)$ is determined to be

$$d_0(x, y) = r(\overline{x} \star y - \overline{y} \star x) + l(y)l(\overline{x}) - l(x)l(\overline{y}) \tag{1.40a}$$

$$= l(y \star \overline{x} - x \star \overline{y}) + r(y)r(\overline{x}) - r(x)r(\overline{y}). \tag{1.40b}$$

We can then redefine the structurable algebra of Allison [A.78] to be a unital normal Lrt. algebra (see [O.05]):

Def.1.7

A pre-structurable algebra A^* is a unital involutive algebra satisfying
(i)

$$d_1(x,y) = l(\overline{y})l(x) - l(\overline{x})l(y), \tag{1.41a}$$

$$d_2(x,y) = r(\overline{y})r(x) - r(\overline{x})r(y), \tag{1.41b}$$

$$\begin{aligned} d_0(x,y) &= r(\overline{x}\star y - \overline{y}\star x) + l(y)l(\overline{x}) - l(x)l(\overline{y}) \\ &= l(y\star\overline{x} - x\star\overline{y}) + r(y)r(\overline{x}) - r(x)r(\overline{y}) \end{aligned} \tag{1.41c}$$

(ii)

$$\overline{d_j(x,y)}(u\star v) = (d_{j+1}(x,y)u)\star v + u\star(d_{j+2}(x,y)v). \tag{1.41d}$$

Moreover if it satisfies the additional condition
(iii)

$$Q(x,y,z) = d_0(x,\overline{y\star z}) + d_1(z,\overline{x\star y}) + d_2(y,\overline{z\star x}) = 0, \tag{1.42}$$

then we call A^* be a structurable algebra. ([K-O.14])

Note that the conjugate algebra A of a pre-structurable or structurable algebra A^* is a pre-normal or normal triality algebra A, respectively, satisfying $xe = ex = \overline{x}$ and $d_0(x,y)z = (\overline{x}y - \overline{y}x)\overline{z} + (\overline{x}z)\overline{y} - (\overline{y}z)\overline{x} = \overline{z}(y\overline{x} - x\overline{y}) + \overline{y}(z\overline{x}) - \overline{x}(z\overline{y})$, since $\overline{x * y} = xy$.

Remark 1.8

If A is a normal triality algebra, then $D(x,y)$ defined by Eq.(1.32) is a derivation satisfying

$$D(x,yz) + D(y,zx) + D(z,xy) = 0 \tag{1.43}$$

in view of Eq.(1.35). In [Kam.95], any algebra A which posseses a derivation $D(x,y) = -D(y,x)$ satisfying Eq.(1.43) has been called a generalized structurable algebra. Therefore, any normal triality algebra is a generalized structurable algebra if $D(x,y)$ is *not* trivial. Note that there exists a triality algebra with $D(x,y) = 0$ identically (see Eq.(2.20)).

Many interesting algebra such as Malcev, structurable, admissible cubic algebra ([E-O.00]) and pseudo-composition algebra [M-O.93] are known to be generalized structurable algebras. (see [Kam.95],[O.05]).

Remark 1.9

We can generalize the idea to super-algebra ([K-O.00]). Let A be Z_2-graded as

$$A = A_{\bar{0}} \oplus A_{\bar{1}}. \tag{1.44}$$

We write for simplicity

$$(-1)^x = (-1)^{grad\ x}, \tag{1.45a}$$

where

$$\text{grad}\ x = \begin{cases} 0, & \text{if} \quad x \in A_{\bar{0}} \\ 1, & \text{if} \quad x \in A_{\bar{1}}. \end{cases} \tag{1.45b}$$

Then, we replace the definition for $d_j(x,y)'$s as

$$d_1(x,y) = (-1)^{xy} R(y)L(x) - R(x)L(y) \tag{1.46a}$$

$$d_2(x,y) = (-1)^{xy} L(y)R(x) - L(x)R(y) \tag{1.46b}$$

while the triality relation Eq.(1.19) must be replaced by

$$d_j(x,y)(uv) = (d_{j+1}(x,y)u)v + (-1)^{(x+y)u} u(d_{j+2}(x,y)v) \tag{1.47}$$

etc. Then, all statements so for given in this section will proceed accordingly.

2. Examples of Normal Triality Algebras

Example 2.1,(Lie and Jordan algebra)

Both Lie and Jordan algebras are normal triality algebras. Writing the bi-linear product of these algebras as xy, we have

$$xy = \varepsilon yx \tag{2.1}$$

for $\varepsilon = +1$ or -1, respectively for Jordan or Lie algebra, so that

$$L(x) = \varepsilon R(x).$$

Setting then

$$d(x,y) := d_0(x,y) = d_1(x,y) = d_2(x,y) = -\varepsilon[L(x), L(y)], \tag{2.2}$$

it is a inner derivation of these algebra, satisfying ([Kam.95])

$$Q(x,y,z) = d(x,yz) + d(y,zx) + d(z,xy) = 0.$$

Moreover Eq.(1.21) is a cosequence of the Jacobi identity for Lie, while it is trivially satisfied for the case of the Jordan algebra.

Moreover, they are involutive with the involution

$$\overline{x} = +\varepsilon x,$$

so that they are also normal Lrt algebra with $x \star y = \overline{xy} = \varepsilon xy$.

Example 2.2,(Symmetric Composition Algebras)

Let A be an algebra with symmetric bi-linear non-degenerate form $< \cdot | \cdot >$ over the field F of charachteristic $\neq 2$. Suppose that we have

$$x(yx) = (xy)x =< x|x > y, \tag{2.3}$$

for $x, y \in A$. Then, A is known as a symmetric composition algebra, since then it satisfies also

$$< xy|xy >=< x|x >< y|y >, \ < xy|z >=< x|yz > . \tag{2.4}$$

Conversely the validity of Eq.(2.4) gives Eq.(2.3) ([O-O.81]). Moreover, a symmetric composition algebra is either a para-Hurwitz algebra or a eight-dimensional pseudo-octonion algebra. ([O-O.81], [O.95])

Here, the para-Hurwitz algebra is the conjugate algebra of the Hurwitz (i.e. unital composition) algebra. Any symmetric composition algebra satisfy the triality relation for the choice of

$$d_0(x,y) = 2\{[L(x), L(y)] - R([x,y])\} \tag{2.5a}$$

or equivalently by

$$d_0(x,y)z = 4\{< x|z > y- < y|z > x\}, \tag{2.5b}$$

as has been noted in ([KMRT.98] and [E.97]), and it is a normal triality algebra ([O.05]).

We also note that the para-Hurwitz algebra has the para-unit e but the pseudo-octonion algebra possesses neither unit nor para-unit.

Example 2.3,(Tensor product)

Let A_1 and A_2 be two independent symmetric composition algebras. Then, their tensor product $A_1 \otimes A_2$ is normal triality algebra with (see [O.05])

$$D_j(x_1 \otimes x_2, y_1 \otimes y_2) :=$$

$$d_j^{(1)}(x_1,y_1)\otimes < x_2|y_2 >_2 \mathrm{id} + < x_1|y_1 >_1 \mathrm{id}\otimes d_j^{(2)}(x_2,y_2), \tag{2.6}$$

for $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$.

As we will show in the next section, this case is relevant for a construction of the so-called Freudenthal's magic square.

Example 2.4

Let A be a normal triality algebra with a order 3 automorphism Φ ($i.,e,$ $\Phi^3 = 1$). Suppose that it also satisfies

$$\Phi d_0(x,y)\Phi^{-1} = d_0(\Phi x, \Phi y), \tag{2.7}$$

which holds automatically if the condition (B) or (C) is valid. If we introduce then a new bi-linear product in the same vector space A by

$$x \circ y = (\Phi x)(\Phi^2 y) \tag{2.8}$$

then the resulting new algebra $A^{(\circ)}$ is a normal triality algebra ([E-O.07]), so that a symmetric composition algebra A is transformed into another symmetric composition algebra $A^{(\circ)}$([E.97]).

As an example, consider the $so(3)$ Lie algebra:

$$e_i e_j = \Sigma_{k=1}^3 \epsilon_{ijk} e_k \quad (i,j = 1,2,3)$$

for a Levi-Civita symbol ϵ_{ijk}. Since it is a Lie algebra, it is a normal symmetric triality algebra by Example 2.1. Moreover, $\Phi \in \mathrm{End}(so(3))$ defined by

$$\Phi : e_1 \to e_2 \to e_3 \to e_1$$

is its order 3 automorphysm. We then calculate the new product to satisfy

$$\begin{aligned} &(1)\ e_1 \circ e_1 = e_1, \quad e_2 \circ e_2 = e_2, \quad e_3 \circ e_3 = e_3 \\ &(2)\ e_1 \circ e_2 = -e_3, \quad e_2 \circ e_3 = -e_1, \quad e_3 \circ e_1 = -e_2 \\ &(3)\ e_2 \circ e_1 = e_1 \circ e_3 = e_3 \circ e_2 = 0 \end{aligned} \tag{2.9}$$

as in [O.05]. This algebra has some interesting property. We intoduce the bi-linear symmetric non-degenerate form $< \circ|\circ >$ by

$$< e_i|e_j >= \delta_{ij} \quad (i,j = 1,2,3). \tag{2.10}$$

Then, it is a normal triality algebra with $d_0(x,y)$ given by

$$d_0(x,y)z =< x|z > y- < y|z > x. \tag{2.11}$$

Moreover, we have

$$(x \circ x) \circ (x \circ x) =< x|x \circ x > x \tag{2.12}$$

so that the 3rd bi-linear product defined by

$$x \cdot y = \frac{1}{2}(x \circ y + y \circ x)$$

gives a 3-dimensional admissible-cubic algebra ([E-O.06]). Moreover for $x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, $(\lambda_j \in F)$, we set

$$t(x) = \lambda_1 + \lambda_2 + \lambda_3, \quad q(x) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

with $f = e_1 + e_2 + e_3$. Then, they satisfy quadratic relation of ([O.06])

(i)

$$f \circ f = 0$$

(ii)

$$x \circ x - t(x)x + q(x)f = 0. \tag{2.13}$$

Example 2.5

Let

$$A = \text{span} < e, f, x_\mu, x^\mu, \ (\mu = 1, 2, \cdots, N) >$$

with the multiplication table of

(1) $ee = e\,, ff = e,\ ef = fe = -f$

(2) $ex_\mu = x_\mu e = x_\mu,\ x^\mu e = ex^\mu = x^\mu$

(3) $fx_\mu = -x_\mu f = x_\mu,\ fx^\mu = -x^\mu f = -x^\mu$

(4) $x_\mu x_\nu = 0 = x^\mu x^\nu$

(5) $x^\mu x_\nu = -2\delta^\mu_\nu(f + e)$

(6) $x_\nu x^\mu = 2\delta^\mu_\nu(f-e)$

for $\mu,\nu = 1,2,\cdots N$. Then A is a normal triality algebra. Note that A possesses a few involution maps:

Involution 1

$\overline{f} = -f$, but $\overline{x} = x$, for $x = e, x^\mu$, and x_μ.

Involution 2

$\overline{x^\mu} = x_\mu,\ \overline{x_\mu} = x^\mu$, but $\overline{x} = x$, for $x = e$, and f.

Involution 3

$\overline{e} = e$, but $\overline{x} = -x$ for $x = f,\ x_\mu$ and x^μ.

The case of the involution 1 is of interest, since then it satisfies $ex = xe = \overline{x}$ so that e is the para-unit of A.Then, its conjugate algebra A^* is structurable.

In section 4, we will show that this algebra is intimately related to the A_4 or S_4 symmetry of the Lie algebra $sl(N),\ (N \geq 4)$.

Example 2.6 (Structurable Algebra)

It is known ([A-F.93]) that any unital involutive alternative or Jordan algebra is structurable. Especially, any unital composition algebra as well as any unital involutive associative algebra is structurable. Moreover some class of Zorn's vector matrix algebras are also structurable. Let B be a involutive algebra over a field F with bi-linear product xy and with a bi-linear form $(\circ|\circ)$, and consider a vector space of form

$$A = \begin{pmatrix} F & B \\ B & F \end{pmatrix}. \tag{2.14}$$

Designating a generic element of A as

$$X = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix},\ (x, y \in B,\ \alpha, \beta \in F) \tag{2.15}$$

we introduce a bi-linear product in A by

$$X_1 \star X_2 =$$

$$\begin{pmatrix} \alpha_1 & x_1 \\ y_1 & \beta_1 \end{pmatrix} \star \begin{pmatrix} \alpha_2 & x_2 \\ y_2 & \beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1\alpha_2 + (x_1|y_2), & \alpha_1 x_2 + \beta_2 x_1 + k y_1 y_2 \\ \alpha_2 y_1 + \beta_1 y_2 + k x_1 x_2, & \beta_1\beta_2 + (y_1|x_2) \end{pmatrix} \tag{2.16}$$

for a constant $k \in F$ and for variables $\alpha_j, \beta_j \in F$ and $x_j, y_j \in B (j = 1, 2)$. Then,

$$X \to \overline{X} = \begin{pmatrix} \beta & \bar{x} \\ \bar{y} & \alpha \end{pmatrix} \tag{2.17}$$

is a involution map of A, provided that $(\circ|\circ)$ satisfies

$$(\bar{x}|y) = (\bar{y}|x), \ (= \text{symmetric in } x \text{ and } y). \tag{2.18}$$

If B is a commutative cubic-admissible algebra over the field F of characteristic $\neq 2$, and $\neq 3$, satisfying

$$x^2x^2 =< x|x^2 > x, \text{with } (x|y) = 3 < x|y >, \tag{2.19}$$

then A is known to be structurable for the choice $k = 2$ ([O.05]).

As an example, consider the case of Dim$B = 1$ with $B = Fb$, where $b \in B$ satisfies

$$bb = b, \ < b|b >= 1.$$

If we set now

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, h = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

then e is the unit element of A so that $e \star x = x \star e = x$, also, by means of $(b|b) = 3 < b|b >= 3$, the multiplication table is given by

$$f \star f = e, \ f \star g = -g \star f = -g, \ f \star h = -h \star f = h,$$

$$g \star g = 2h, \ h \star h = 2g, \ g \star h = \frac{3}{2}(e - f), \ h \star g = \frac{3}{2}(e + f) \tag{2.20}$$

as in [O,06]. A peculiar aspect of this algebra is that we have $D(x, y) = 0$ identically. Further, A admits few involutions:

(1) $\bar{f} = -f$, but $\bar{x} = x$ for $x = e, g$, and h, corresponding to Eq.(2.17).

(2) $\overline{g} = h, \overline{h} = g$, but $\overline{x} = x$ for $x = e$ and f.

(3) Let $\omega \in F$ to satisfy $\omega^3 = 1, \omega \neq 1$. $\overline{h} = \omega h, \overline{g} = \omega^2 g, \overline{f} = -f, \overline{e} = e$.

On the other side, if B is an anti-commutative algebra, then A is an alternative algebra, provided that we have

$$x(yz) = (x|y)z - (x|z)y,$$
$$(x|yz) = (y|zx) = (z|xy)$$

with $\overline{x} = -x$ and $k = 1$. This case yields the octonion algebra as well as a unconventional six-dimensional degenerate composition algebra associated with a five-dimentional Malcev algebra [K-O,14], although we will not go into its details here.

3 Lie Algebra satisfying Triality

Let A be a pre-normal triality algebra as in Def.1.2, and consider linear maps:

$$\rho_j : A \to V, \text{and } T_j : A \otimes A \to V \tag{3.1}$$

for $j = 0, 1, 2$, where V is an unspecified algebra with skew symmetric bilinear product $[\circ, \circ]$. We set now

$$T(A, A) = \text{span} < T_j(x, y), \forall j = 0, 1, 2, \forall x, y \in A > \tag{3.2}$$

and

$$L(A) = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A, A). \tag{3.3}$$

Following [A-F,93], let (i, j, k) be a cyclic permutation of indices $(0, 1, 2)$, and assume the following anti-communtative multiplication relations:

(1)

$$[\rho_i(x), \rho_i(y)] = -[\rho_i(y), \rho_i(x)] = \gamma_j \gamma_k^{-1} T_{3-i}(x, y) \tag{3.4a}$$

(2)

$$[\rho_i(x), \rho_j(y)] = -[\rho_j(y), \rho_i(x)] = -\gamma_j \gamma_i^{-1} \rho_k(xy) \tag{3.4b}$$

(3)

$$[T_l(x, y), \rho_j(z)] = -[\rho_j(z), T_l(x, y)] = \rho_j(d_{l+j}(x, y)z) \tag{3.4c}$$

(4)

$$\begin{aligned}[T_l(u, v), T_m(x, y)] &= T_m(d_{l-m}(u, v)x, y) + T_m(x, d_{l-m}(u, v)y) \\ &= -T_l(d_{m-l}(x, y)u, v) - T_l(u, d_{m-l}(x, y)v)\end{aligned} \tag{3.4d}$$

for $l, m = 0, 1, 2$. Here, $\gamma_j \in F$ are some non-zero constants. We introduce the Jacobian in $L(A)$ by

$$J(X, Y, Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \tag{3.5}$$

for $X, Y, Z \in L(A)$.

We first prove.

Lemma 3.1

$T(A, A)$ and $T_j(A, A)$ for $j = 0, 1, 2$ are Lie algebras. Also $T_j(A, A)$ is an ideal of $T(A, A)$.

Proof

We calculate now for any $j, k, l = 0, 1, 2$,

$$\begin{aligned}
&[[T_j(u,v), T_k(x,y)], T_l(z,w)] = -[T_l(z,w), [T_j(u,v), T_k(x,y)]] \\
&= -[T_l(z,w), T_k(d_{j-k}(u,v)x, y) + T_k(x, d_{j-k}(u,v)y)] \\
&= -T_k(d_{l-k}(z,w)d_{j-k}(u,v)x, y) - T_k(d_{j-k}(u,v)x, d_{l-k}(u,v)y) \\
&+ T_k(d_{l-k}(z,w)x, d_{j-k}(u,v)y) - T_k(x, d_{l-k}(z,w)d_{j-k}(u,v)y)
\end{aligned}$$

$$\begin{aligned}
&[[T_l(z,w), T_j(u,v)], T_k(x,y)] = -[[T_j(u,v), T_l(z,w)], T_k(x,y)] \\
&= -[T_l(d_{j-l}(u,v)z, w) + T_l(z, d_{j-l}(u,v)w), T_k(x,y)] \\
&= -T_k(d_{l-k}(d_{j-l}(u,v)z, w)x, y) - T_k(x, d_{l-k}(d_{j-l}(u,v)z, w)y) \\
&- T_k(d_{l-k}(z, d_{j-l}(u,v)w)x, y) - T_k(x, d_{l-k}(z, d_{j-l}(u,v)w)y)
\end{aligned}$$

and

$$\begin{aligned}
&[[T_k(x,y), T_l(z,w)], T_j(u,v)] = [T_j(u,v), [T_l(z,w), T_k(x,y)]] \\
&= [T_j(u,v), T_k(d_{l-k}(z,w)x, y) + T_k(x, d_{l-k}(z,w)y)] \\
&= T_k(d_{j-k}(u,v)d_{l-k}(z,w)x, y) + T_k(d_{l-k}(z,w)x, d_{j-k}(u,v)y) \\
&+ T_k(d_{j-k}(u,v)x, d_{l-k}(z,w)y) + T_k(x, d_{j-k}(u,v)d_{l-k}(z,w)y).
\end{aligned}$$

Adding these three relations we find

$$J(T_j(u,v), T_k(x,y), T_l(z,w)) = T_k(\lambda x, y) + T_k(x, \lambda y)$$

where λ is given by

$$\begin{aligned}
\lambda = [d_{j-k}(u,v), d_{l-k}(z,w)] \\
- d_{l-k}(d_{j-l}(u,v)z, w) - d_{l-k}(z, d_{j-l}(u,v)w) = 0
\end{aligned}$$

by the triality Lie relation Eq.(1.20).□

We next set

$$J(x,y,z) := J(\rho_0(x),\rho_1(y),\rho_2(z)). \tag{3.6}$$

Lemma 3.2

We have

$$J(x,y,z) = T_0(x,yz) + T_1(z,xy) + T_2(y,zx) \tag{3.7}$$

which satisfies

$$[J(x,y,z),\rho_i(w)] = \rho_i(Q(z,x,y)w) \tag{3.8a}$$

$$[J(x,y,z),T_l(u,v)] = T_l(Q_l(z,x,y)u,v) + T_l(u,Q_l(z,x,y)v) \tag{3.8b}$$

where we have set

$$Q_l(z,x,y) = d_{-l}(z,xy) + d_{1-l}(y,zx) + d_{2-l}(x,yz). \tag{3.8c}$$

Note the $Q_0(z,x,y) = Q(z,x,y),\ Q_1(z,x,y) = Q(y,z,x)$ etc.

Proof

These are straightforward results of Eqs.(3.4).□

Proposition 3.3

Let A be a prenormal triality algebra. Then we have $J(X,Y,Z) = 0$ for X,Y and Z being any one of forms $\rho_i(z)$ or $T_j(x,y)$ except for $J(\rho_i(x),\rho_j(y),\rho_k(z))$ or $J(\rho_0(x),\rho_1(y),\rho_2(z))$.

Proof

(1) We calculate

$$\begin{aligned}&[[\rho_i(x),\rho_i(y)],\rho_i(z)] = [\gamma_j\gamma_k^{-1}T_{3-i}(x,y),\rho_i(z)]\\ &= \gamma_j\gamma_k^{-1}\rho_i(d_3(x,y)z) = \gamma_j\gamma_k^{-1}\rho_i(d_0(x,y)z)\end{aligned}$$

so that we have

$$J(\rho_i(x),\rho_i(y),\rho_i(z)) = \gamma_j\gamma_k^{-1}\rho_i(w)$$

with

$$w = d_0(x,y)z + d_0(y,z)x + d_0(z,x)y = 0$$

by Eq.(1.21). Thus, we have $J(\rho_i(x), \rho_i(y), \rho_i(z)) = 0$.

(2) We similarly compute

$$[[\rho_i(x), \rho_i(y)], \rho_j(z)] = [\gamma_j\gamma_k^{-1}T_{3-i}(x,y), \rho_j(z)]$$
$$= \gamma_j\gamma_k^{-1}\rho_j(d_{3-i+j}(x,y)) = \gamma_j\gamma_k^{-1}\rho_j(d_1(x,y)z)$$

when we note $j - i = 1(\bmod 3)$ since (i,j,k) is a cyclic pertation of $(0,1,2)$. Further we note

$$[[\rho_i(y), \rho_j(z)], \rho_i(x)] = [-\gamma_i\gamma_i^{-1}\rho_k(yz), \rho_i(x)]$$
$$= (-\gamma_j\gamma_i^{-1})(-\gamma_i\gamma_k^{-1})\rho_j((yz)x) = \gamma_j\gamma_k^{-1}\rho_j((yz)x)$$

so that

$$J(\rho_i(x), \rho_i(y), \rho_j(z)) = \gamma_j\gamma_k^{-1}\rho_j(w)$$

with

$$w = d_1(x,y)z + (yz)x - (xz)y = \{d_1(x,y) + R(x)L(y) - R(y)L(x)\}z = 0$$

by Eq.(1.18b). This shows $J(\rho_i(x), \rho_i(y), \rho_j(z)) = 0$.

(3) We analogously compute

$$[[\rho_i(x), \rho_i(y)], \rho_k(z)] = [\gamma_j\gamma_k^{-1}T_{3-i}(x,y), \rho_k(z)]$$
$$= \gamma_j\gamma_k^{-1}\rho_k(d_{3-i+k}(x,y)z) = \gamma_j\gamma_k^{-1}(d_2(x,y)z)$$

since $k - i = 2(\bmod 3)$, while

$$[[\rho_i(y), \rho_k(z)], \rho_i(x)] = -[[\rho_k(z), \rho_i(y)], \rho_i(x)]$$
$$= \gamma_i\gamma_k^{-1}[\rho_j(zy), \rho_i(x)] = -\gamma_i\gamma_k^{-1}[\rho_i(x), \rho_j(zy)]$$
$$= (-\gamma_i\gamma_k^{-1})(-\gamma_j\gamma_i^{-1})\rho_k(x(zy)) = \gamma_j\gamma_k^{-1}\rho_k(x(zy)).$$

In this way, we obtain

$$J(\rho_i(x), \rho_i(y), \rho_k(z)) = \gamma_j\gamma_k^{-1}\rho_k(w)$$

where

$$w = d_2(x,y)z + x(zy) - y(zx) = \{d_2(x,y) + L(x)R(y) - L(y)R(x)\}z = 0,$$

by Eq.(1.18c).

(4) However,

$$[[\rho_i(x),\rho_j(y)],\rho_k(z)] = [-\gamma_j\gamma_i^{-1}\rho_k(xy),\rho_k(z)] = -\gamma_j\gamma_i^{-1}\gamma_i\gamma_j^{-1}T_{3-k}(xy,z)$$
$$= -T_{3-k}(xy,z) = T_{3-k}(z,xy)$$

so that

$$J(\rho_i(x),\rho_j(y),\rho_k(z)) = T_{3-k}(z,xy) + T_{3-i}(x,yz) + T_{3-i}(y,zx) \qquad (3.6)'$$

which gives Eq.(3.7) for $i=0, j=1$ and $k=2$.

(5) We similarly compute

$$[[\rho_i(x),\rho_i(y)],T_l(u,v)] = [\gamma_j\gamma_k^{-1}T_{3-i}(x,y),T_l(u,v)]$$
$$= -\gamma_j\gamma_k^{-1}\{T_{3-i}(d_{l+i}(u,v)x,y) + T_{3-i}(x,d_{l+i}(u,v)y)\}$$

and

$$[[\rho_i(y),T_l(u,v)],\rho_i(x)] = [-\rho_i(d_{i+l}(u,v)y),\rho_i(x)]$$
$$= \gamma_j\gamma_k^{-1}T_{3-i}(x,d_{i+l}(u,v)y).$$

Then, we see $J(\rho_i(x),\rho_i(y),T_l(u,v)) = 0$.

(6) Moreover, We note

$$[[\rho_i(x),\rho_j(y)],T_l(u,v)] = [-\gamma_j\gamma_i^{-1}\rho_k(xy),T_l(u,v)] = \gamma_j\gamma_i^{-1}\rho_k(d_{l+k}(u,v)(xy)),$$

$$[[\rho_j(y),T_l(u,v)],\rho_i(x)] = [-\rho_j(d_{j+l}(u,v)y),\rho_i(x)] = -\gamma_j\gamma_i^{-1}\rho_k(x\{d_{j+l}(u,v)y\}),$$

and

$$[[T_l(u,v),\rho_i(x)],\rho_j(y)] = [\rho_i(d_{l+i}(u,v)x),\rho_j(y)] = -\gamma_j\gamma_i^{-1}\rho_k(\{d_{l+i}(u,v)x\}y).$$

Thus we obtain

$$J(\rho_i(x),\rho_j(y),T_l(u,v)) = \gamma_j\gamma_i^{-1}\rho_k(w),$$

with

$$w = d_{l+k}(u,v)(xy) - x\{d_{j+l}(u,v)y\} - \{d_{i+l}(u,v)x\}y.$$

But then $w=0$ by the triality relation Eq.(1.19).

(7) We similarly find

$$J(\rho_k(x),T_l(u,v),T_m(x,y)) = \rho_k(\lambda z)$$

with

$$\lambda = [d_{k+m}(x,y), d_{k+l}(u,v)] + \\ d_{k+m}(d_{l-m}(u,v)x,y) + d_{k+m}(x, d_{l-m}(u,v)y) = 0$$

by Eq.(1.20).

(8) We have already noted in Lemma 3.1 that we have for any $j,k,l = 0,1,2$,

$$J(T_j(u,v), T_k(x,y), T_l(z,w)) = 0. \square$$

In this connection, we consider

Condition (D)

Suppose that we have $\rho_i(x) = 0$ for some $x \in A$ and for some value of $i = 0,1,2$. We then have $x = 0$.

Corollary 3.4

Let A be a pre-normal triality algebra. If we have

$$J(x,y,z) = T_0(x,yz) + T_1(z,xy) + T_2(y,zx) = 0, \tag{3.9}$$

then $L(A)$ is a Lie algebra. Moreover, if the condition (D) holds, then A is a normal triality algebra. Conversely, if $L(A)$ is a Lie algebra and if the condition (D) holds, then A is a normal triality algebra with the validity of Eq.(3.9).

Proof

This follows from Lemma 3.2 as well as the proof given in Proposition 3.3. □

If we do *not* assume the validity of Eq.(3.9), we set

$$J = \text{span} < J(x,y,z), x,y,z \in A > . \tag{3.10}$$

If A is a normal triality algebra, then Lemma 3.2 implies that $J(x,y,z)$ are center elements of A, since $Q(x,y,z) = 0$. Then, we find

Theorem 3.5

Let A be a normal triality algebra. Then, the quotient algebra $\tilde{L} = L/J$ is a Lie algebra.

Hereafter in this section, we assume A to be a normal triality algebra unless it is stated otherwise. Then, in view of Theorem 3.5 we can effectively assume the validity of Eq.(3.9). As a matter of fact, if we identify $T_j(x,y)$ with the triple

$$T_j(x,y) = (d_j(x,y), d_{j+1}(x,y), d_{j+2}(x,y)) \quad (j = 0,1,2) \tag{3.11}$$

(see i.e. [A-F,93], and [E.04]), then we find

$$T_0(x,yz) + T_1(z,xy) + T_2(y,zx) =$$

$$(Q(x,y,z), Q(y,z,x), Q(z,x,y)) = 0.$$

Moreover, they will yield $T_0(x,y) = T_1(x,y) = T_2(x,y)$ if we have $d_0(x,y) = d_1(x,y) = d_2(x,y)$ as in the case of Lie and Jordan algebra (see Example 2.1). This can be also justified without assuming Eq.(3.11) as follows: If $d_0(x,y) = d_1(x,y) = d_2(x,y)$, then we see from Eq.(3.4) that the differences $T_i(x,y) - T_j(x,y)$ for $i \neq j$ are center element of $L(A)$ so that we can effectively set $T_i(x,y) = T_j(x,y)$. This fact will be assumed and used in the next section for S_4-symmetry of the Lie algebra $so(N)$.

We will assume also for simplicity the validity of Eq.(3.9) or Eq.(3.11) hereafter unless it is stated otherwise.

For the case of A^* being a structurable algebra, we need simply replace $\rho_k(xy)$ in Eq.(3.4b) by

$$\rho_k(xy) \to \rho_k(\overline{\overline{x} \star \overline{y}}) = \rho_k(\overline{y} \star \overline{x}) \tag{3.12a}$$

and Eq.(3.9) by

$$J(x,y,z) = T_0(x, \overline{\overline{y} \star \overline{z}}) + T_1(z, \overline{\overline{x} \star \overline{y}}) + T_2(y, \overline{\overline{z} \star \overline{x}}) = 0 \tag{3.12b}$$

according to Eq.(1.11) for the Lie algebra $L(A)$.

Now,a special choice of $\gamma_0 = \gamma_1 = \gamma_2 = 1$ for constants γ_j in Eqs.(3.4) is of a particular interest, since the Lie algebra $L(A)$ will admit then an alternative group (or equivalently tetrahedral group T_4) A_4 as automorphism.

First , $L(A)$ is clearly invariant under actions of a cyclic group Z_3 generated by $\phi \in \text{End}\, L(A)$ given by

$$\rho_i(x) \to \rho_{i+1}(x),\ T_i(x,y) \to T_{i-1}(x,y). \tag{3.13}$$

Next, let $\tau_\mu \in \text{End } L(A)$ for $\mu = 1, 2, 3$ be defined by

$$\begin{aligned} &\tau_1 : \rho_1(x) \to \rho_1(x),\ \rho_2(x) \to -\rho_2(x),\ \rho_3(x) \to -\rho_3(x), \\ &\tau_2 : \rho_1(x) \to -\rho_1(x),\ \rho_2(x) \to \rho_2(x),\ \rho_3(x) \to -\rho_3(x) \\ &\tau_3 : \rho_1(x) \to -\rho_1(x),\ \rho_2(x) \to -\rho_2(x),\ \rho_3(x) \to \rho_3(x) \end{aligned} \tag{3.14}$$

while $T_j(x, y)$ for $j = 0, 1, 2$ remains unchanged by actions of τ_μ. Then, $L(A)$ is also invariant under τ_μ. Moreover, we note

$$\tau_\mu \tau_\nu = \tau_\nu \tau_\mu,\ \tau_\mu \tau_\mu = 1,\ \tau_1 \tau_2 \tau_3 = 1,\ (\mu, \nu = 1, 2, 3) \tag{3.15}$$

so that $(1, \tau_1, \tau_2, \tau_3)$ is isomorphic to the Klein's 4-group K_4.

Further, we see

$$\phi \tau_\mu \phi^{-1} = \tau_{\mu+1}\ (\mu = 1, 2, 3) \tag{3.16}$$

with $\tau_4 = \tau_1$. Since Z_3 and K_4 generate the alternative group A_4 (an equivalently the tetrahedral group T_4), the Lie algebra $L(A)$ is invariant under A_4.

If A is involutive with the involution map $x \to \overline{x}$ in addition, then $\tau \in \text{End } L(A)$ given by

$$\begin{aligned} &\tau : \rho_1(x) \leftrightarrow -\rho_2(\overline{x}),\ \rho_3(x) \to -\rho_3(\overline{x}), \\ &T_1(x, y) \leftrightarrow T_2(\overline{x}, \overline{y}),\ T_3(x, y) \to T_3(\overline{x}, \overline{y}) \end{aligned} \tag{3.17}$$

also defines an automorphsm of $L(A)$ satisfying

$$\tau^2 = 1,\ \tau \tau_1 \tau^{-1} = \tau_2,\ \tau \tau_3 \tau^{-1} = \tau_3,\ \phi \tau \phi = \tau. \tag{3.18}$$

Then, τ and A_4 generate the symmetric group S_4 with identifications of

$$\tau_1 = (2, 3)(1, 4),\ \tau_2 = (3, 1)(2, 4),\ \tau_3 = (1, 2)(3, 4),\ \phi = (1, 2, 3),\ \tau = (1, 2) \tag{3.19}$$

in the standard notation for symmetric group.

Regarding $L(A)$ as a A_4-module, the triple $(\rho_0(x), \rho_1(x), \rho_2(x))$ for any $x \in A$ realizes then a 3-dimensional irreducible module of A_4. For $T(A, A)$, we assume for simplicity, that the underlying field F is of charachtericsitc $\neq 2$, and $\neq 3$. If F contains $\omega \in F$ satisfying $\omega^3 = 1$ but $\omega \neq 1$, then $T(A, A)$ is a direct sum of three inequivalent one-dimensional modules given by

$$\varphi_n(x, y) = T_0(x, y) + \omega^n T_1(x, y) + \omega^{2n} T_2(x, y)$$

for $n = 0, 1, 2$. However, if F does *not* contains such $\omega \in F$, then,

$$\varphi_0(x, y) = T_0(x, y) + T_1(x, y) + T_2(x, y)$$

is the trivial module of A_4 and $(\theta_1(x, y), \theta_2(x, y))$ defined by

$$\begin{aligned} \theta_1(x, y) &= T_1(x, y) + T_2(x, y) - 2T_0(x, y), \\ \theta_2(x, y) &= T_1(x, y) - T_2(x, y) \end{aligned}$$

represents two-dimensional irreducible module of A_4.

The case of the S_4-symmetry is slightly more involved, since we have to take account of the action of $\tau = (1, 2)$ in addition. In that case, depending upon $\overline{x} = x$ or $\overline{x} = -x$, the triple $(\rho_0(x), \rho_1(x), \rho_2(x))$, represents two inequivalent 3-dimentional modules of S_4, while for $T_j(A, A)$, we have to consider 4 cases of $\overline{x} = \pm x$ and $\overline{y} = \pm y$ or $\overline{y} = \mp y$ to find two inequivalent two-dimensional modules $(\theta_1(x, y), \theta_2(x, y))$ and one-dimentional modules $\varphi_0(x, y)$ of S_4.

Returning to the structure of $L(A)$, we set

$$L_j(A) = \rho_j(A) \oplus T_{3-j}(A, A), \;\; (j = 0, 1, 2). \tag{3.20}$$

We have then

$$L(A) = L_0(A) + L_1(A) + L_2(A). \tag{3.21}$$

As we see from Eqs.(3.4), $L_j(A)$ $(j = 0, 1, 2)$ are sub-Lie algebras of $L(A)$, while $T_{3-j}(A, A)$ is a sub-Lie algebra of $L_j(A)$. Moreover, under action of Z_3, we have

$$\begin{aligned} \phi : \;\; & L_0(A) \to L_1(A) \to L_2(A) \to L_0(A) \\ & T_0(A, A) \to T_2(A, A) \to T_1(A, A) \to T_0(A, A) \end{aligned} \tag{3.22}$$

while they transform among themselves under action of the Klein's 4-group K_4.

It may be instructive to depict $L(A)$ as in Fig.1(Appendix), exhibiting the triality.

As illustration, let us examine specific cases of Examples given in section 2, assuming the underlying field F to be algebraically closed and of charateristic $\neq 2, \neq 3$ for simplicity.

Example 3.6(Lie algebra G_2)

The 4-dimensional structurable algebra $A^* =< e, f, g, h >$ given by Eq.(2.20) leads to

$$L(A) = G_2, \; L_j(A) = A_1 \oplus A_1 (j = 0, 1, 2) \text{ and}$$
$$T_j(A, A) = gl(1) \oplus gl(1). \; (j = 0, 1, 2)$$

as in [O.06].

Example 3.7(magic Square)

Let $A = A_1 \otimes A_2$ be the tensor product algebra of two independest symmetric composition algebra as in Example 2.3. Then, A is also a normal triality algebra, and we can construct Lie algebras by Theorem 3.5. Following [E.04] and [E.06], this leads to the Freudenthal's magic square for the Lie algebra $L(A)$ as in Fig.2 (see also [Ba-S.03]):

$Dim\ A_1 \setminus Dim\ A_2$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

Fig.2:Magic Square

If A_1 is a para-octonion or pseudo-octonion algebra, and if we choose Dim $A_2 = 1$, then the resulting Lie algebras are

$$L(A) = F_4, \; L_j(A) = B_4, \; T_j(A, A) = D_4 \; (j = 0, 1, 2),$$

corresponding to the classical triality case of A_1^* being octonion algebra.

For other case of Dim A_1 = Dim A_2 = 8, where A_1 and A_2 are either para-octonion or pseudo-octonion algebra, we obtain

$$L(A) = E_8, \; L_j(A) = D_8, \; T_j(A, A) = D_4 \oplus D_4.$$

Example 3.8(Zorn's Vector Matrix Algebra)

Let us consider Example 2.6 again where the algebra B is now the 27-dimensional cubic-admissible algebra associated with the Albert algebra.

In that case, it is known ([Kan 73] and [Kam 89]) that $L(A)$ is also the Lie algebra E_8. However, we have

$$L_j(A) = E_7 \oplus A_1 \text{ and } T_j(A, A) = E_6 \oplus gl(1) \oplus gl(1)$$

in contrast to the previous case of example 3.7.

Remark 3.9

In ending this section, we note that any finite dimensional normal triality algebra satisfying the condition (D) may be identified with some symmetric space. For example, Eq.(3.4) implies

$$\begin{aligned}
&[\rho_0(x), \rho_0(y)] = T_0(x, y), \\
&[T_0(x, y), \rho_0(z)] = \rho_0(d_0(x, y)z), \\
&[T_0(u, v), T_0(x, y)] = T_0(d_0(u, v)x, y) + T_0(x, d_0(u, v)y)
\end{aligned}$$

for a Lie algebra $L_0(A)$, so that we may identify $\rho_0(A)$ with the symmetric space

$$L_0(A)/T_0(A, A).$$

Moreover, if the condition (D) for $j = 0$ is satisfied, then $\rho_0(x) = 0$ for some $x \in A$ implies $x = 0$, so that $A \to \rho_0(A)$ is one-to-one map. Hence, we can identify A with the symmetric space.

4. Lie algebras satisfying tetrahedral symmetry

In the previous section, we have seen that we can construct a A_4-invariant Lie algebra out of a normal triality algebra. We will show in this section that the converse statement holds valid also.

Let V be an algebra over a filed F of charachteristic $\neq 2$, endowed with a group homomorphism

$$A_4 \to \text{Auto}(V). \tag{4.1}$$

Let ϕ and $\tau_\mu (\mu = 1, 2, 3) \in A_4$ as in Eq.(3.19), satisfying relations Eqs.(3.15), with $\phi^3 = 1$. Then, V can be decompsed by actions of the Klein's 4-group $K_4 = \{1, \tau_1, \tau_2, \tau_3\}$ into a direct sum

$$V = t \oplus g_1 \oplus g_2 \oplus g_3 \tag{4.2}$$

where

$$t = \{x \in V;\ \tau_1(x) = \tau_2(x) = \tau_3(x) = x\} \tag{4.3a}$$

$$g_1 = \{x \in V;\ \tau_1(x) = x,\ \tau_2(x) = \tau_3(x) = -x\} \tag{4.3b}$$

$$g_2 = \{x \in V;\ \tau_2(x) = x,\ \tau_1(x) = \tau_3(x) = -x\} \tag{4.3c}$$

$$g_3 = \{x \in V;\ \tau_3(x) = x,\ \tau_1(x) = \tau_2(x) = -x\}. \tag{4.3d}$$

We then have

Lemma 4.1

(1)

$$\phi(g_i) = g_{i+1}(\text{with } g_4 = g_1) \tag{4.4a}$$
$$\phi(t) = t$$

(2)

$$tt \subset t, \text{ so that } t \text{ is a subalgebra of } V \tag{4, 4b}$$

(3)

$$tg_i \subset g_i, \text{ and } g_i t \subset g_i \tag{4.4c}$$

(4)

$$g_i g_i \subset t \tag{4.4d}$$

(5) If (i, j, k) is a cyclic permutation of indices $(1, 2, 3)$, then

$$g_i g_j \subset g_k, \text{ and } g_j g_i \subset g_k. \tag{4.4e}$$

Proof

Noting $\phi\tau_\mu = \tau_{\mu+1}\phi$ (with $\tau_4 = \tau_1$) by Eq.(3.16), we obtain

$$\phi(g_i) \subseteq g_{i+1}, \text{ and } \phi(t) \subseteq t.$$

For example, if $x \in g_1$, then we calculate

$$\tau_2\phi x = \phi\tau_1 x = \phi x,\ \tau_3\phi x = \phi\tau_2 x = -\phi x,\ \tau_1\phi x = \phi\tau_3 x = -\phi x$$

which gives $\phi(g_1) \subseteq g_2$. Then we calculate

$$g_i = \phi^3(g_i) \subseteq \phi^2(g_{i+1}) \subseteq \phi(g_{i+2}) \subseteq g_i,$$

which yields $\phi(g_{i+2}) = g_i$, i.e. Eq.(4.4a).

The rest of relations in Eq.(4.4) can be similarly verified, when we note

$$\tau_\mu(xy) = (\tau_\mu x)(\tau_\mu y)$$

for $x, y \in V$. □

Remark 4.2

Setting

$$V_j = t \oplus g_j \ (j = 1, 2, 3), \tag{4.5}$$

we have

$$V = V_1 + V_2 + V_3 \tag{4.6}$$

and we may depict the situation as in Fig.3(Appendix).

Note that V_j $(j = 1, 2, 3)$ are sub-algebras of V.

Example 4.3

Let V be the Cayley algebra with the basis $< e_0, e_1, e_2, \cdots, e_7 >$ with the unit element $e = e_0$, satisfying the multiplication table of

$$e_i e_j = -\delta_{ij} e + \sum_{k=1}^{7} f_{ijk} e_k$$

for $i, j = 1, 2, \cdots, 7$, when f_{ijk} is the totally anti-symmetric constants with values $1, 0, -1$. Moreover, $f_{ijk} = 1$ are possible only for $i, j, k = 123, 516, 624, 435, 174, 376, 275$ with their cyclic permutations. We introduce a self-dual tensor $f_{\mu\nu}$ for $\mu, \nu = 1, 2, 3, 4$ satisfying

$$f_{\mu\nu} = -f_{\nu\mu} = {}^*f_{\mu\nu} = \frac{1}{2} \sum_{\alpha,\beta=1}^{4} \varepsilon_{\mu\nu\alpha\beta} f_{\alpha\beta}$$

(see [O.95]) by

$$e_1 = f_{23} = f_{14}, \ e_2 = f_{31} = f_{24}, \ e_3 = f_{12} = f_{34}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the 4-dimensional Levi-Civita symbol with $\varepsilon_{1234} = 1$.

Moreover,setting

$$a_1 = e_4, \ a_2 = e_5, \ a_3 = e_6, \ a_4 = e_7,$$

then a_μ and $f_{\mu\nu}$ for $\mu, \nu = 1, 2, 3, 4$ satisfy

$$\begin{aligned} a_\mu a_\nu &= -f_{\mu\nu} - \delta_{\mu\nu} e, \\ f_{\mu\nu} a_\lambda &= -a_\lambda f_{\mu\nu} = -\delta_{\mu\lambda} a_\nu + \delta_{\nu\lambda} a_\mu - \textstyle\sum_{\alpha=1}^{4} \varepsilon_{\mu\nu\lambda\alpha} a_\alpha, \\ f_{\mu\nu} f_{\alpha\beta} &= -\delta_{\nu\alpha} f_{\mu\beta} + \delta_{\nu\beta} f_{\mu\alpha} - \delta_{\mu\beta} f_{\nu\alpha} + \delta_{\mu\alpha} f_{\nu\beta} \\ &\quad -(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} + \varepsilon_{\mu\nu\alpha\beta}) e, \end{aligned}$$

which are clearly invariant under any even-permutations of indices 1, 2, 3, and 4, i.e. under the alternative group A_4. We then find

$$\begin{aligned} g_1 &= \{e_2, e_4 - e_5 - e_6 + e_7\}, \\ g_2 &= \{e_3, e_4 - e_5 + e_6 - e_7\}, \\ g_3 &= \{e_1, e_4 + e_5 - e_6 - e_7\}, \\ t &= \{e, e_4 + e_5 + e_6 + e_7\}, \end{aligned}$$

by Eqs.(4.3). Note that $V_j = t \oplus g_j$ $(j = 1, 2, 3)$ are then quaternion subalgebras of the Cayley algebra.

Actually, the Cayley algebra is invariant under S_4, if we define $\tau = (1, 2)$ by

$$\tau : e_1 \to -e_1,\ e_2 \to -e_3 \to e_2,$$

and

$$e_\mu \to \frac{1}{2}(e_4 + e_5 + e_6 + e_7) - \widetilde{e}_\mu, \text{for } \mu = 4, 5, 6, 7,$$

where $\widetilde{e}_4 = e_5, \widetilde{e}_5 = e_4, \widetilde{e}_6 = e_6,$ *and* $\widetilde{e}_7 = e_7$.

Further, any split Cayley algebra is also invariant under S_4. This fact has been used in [E-O.08] to show that all exceptional Lie algebras are S_4-invariant.

However, the most interesting case is obtained, when V is a Lie algebra, as we see from the following Theorem ([E-O.07]). We identify $V = L$ with $g_i g_j \to [g_i, g_j]$ in what follows.

Theorem 4.4

Let L be a Lie algebra over the field F of charachteristic $\neq 2$, which is invariant under the action of the alternative group A_4. Then, there exists a normal triality algebra A such that L is written as a direct sum of

$$L = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus t \qquad (4.7)$$

of some vector spaces $\rho_j(A)$ and a sub-Lie algebra t of L. Moreover, there exits a sub-Lie algebra $T(A, A)$ of t such that

$$\tilde{L} = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A, A) \tag{4.8}$$

is a A_4-invariant ideal of L, which coincides with the Lie algebra constructed in the previous section in terms of the normal triality algebra A, satisfying Eq.(3.4) for $\gamma_0 = \gamma_1 = \gamma_2 = 1$ as well as Eq.(3.9),i.e,

$$T_0(x, yz) + T_1(z, xy) + T_2(y, zx) = 0. \tag{4.9}$$

Further, if L is invariant under a larger group S_4, then A is involutive with a involutive map $x \to \overline{x}$.

Proof

We identify A with g_3 in Eq.(4.3d),i.e,

$$A = \{x \in L;\ \tau_3(x) = x,\ \tau_1(x) = \tau_2(x) = -x\} \tag{4.10}$$

and write

$$\rho_3(x) = \rho_0(x) = x,\ \text{if } x \in A \tag{4.11a}$$

and set

$$\rho_1(x) = \phi x,\ \rho_2(x) = \phi^2 x,\ \text{for } x \in A \tag{4.11b}$$

so that we have

$$\phi \rho_j(x) = \rho_{j+1}(x) \tag{4.12}$$

for any $x \in A$ and for any $j = 1, 2, 3$.

We note that the condition (D) of the previous section is automotically satisfied that if $\rho_j(x) = 0$ for some $x \in A$ and for some $j = 0, 1, 2$, then $x = 0$ in view of Eqs.(4.11).

Next, since $[g_1, g_2] \subseteq g_3$ by Eq.(4.4e) of Lemma 4.1, we can introduce a bi-linear product xy in A by

$$[\rho_1(x), \rho_2(y)] := -\rho_3(xy). \tag{4.13}$$

Applying $\phi \in Z_3$ to this relation, and noting Eq.(4.12), this yield

$$[\rho_i(x), \rho_j(y)] := -\rho_k(xy) \tag{4.14}$$

for any cyclic permutation (i, j, k) of indices $(1, 2, 3)$. This reproduces Eq.(3.4b) for $\gamma_0 = \gamma_1 = \gamma_2 = 1$. Similarly, $[g_j, g_j] \subseteq t$ by Lemma 4.1 and we define $T_j(x, y) \in t$ by

$$[\rho_j(x), \rho_j(y)] := T_{3-j}(x, y). \tag{4.15}$$

Applying ϕ to this relation, it gives

$$\phi T_{3-j}(x, y) = T_{3-(j+1)}(x, y) \tag{4.16}$$

since

$$\begin{aligned} \phi T_{3-j}(x, y) &= \phi[\rho_j(x), \rho_j(y)] = [\phi\rho_j(x), \phi\rho_j(y)] = [\rho_{j+1}(x), \rho_{j+1}(y)] \\ &= T_{3-(j+1)}(x, y). \end{aligned}$$

Analagously, $[t, g_k] \subseteq g_k$ implies that we can define

$$t_{j,k} : A \otimes A \to \text{End}A, \ \ (j, k = 0, 1, 2)$$

by

$$[T_{3-j}(x, y), \rho_k(z)] =: \rho_k(t_{j,k}(x, y)z). \tag{4.17}$$

Operating ϕ to this relation, and noting Eq.(4.16), we calculate

$$[T_{3-(j+1)}(x, y), \rho_{k+1}(z)] = \rho_{k+1}(t_{j,k}(xy)z)$$

or

$$\rho_{k+1}(t_{k+1,j+1}(x, y)z) = \rho_{k+1}(t_{j,k}(x, y)z),$$

which implies

$$t_{k+1,j+1}(x, y)z = t_{j,k}(x, y)z \tag{4.18}$$

because of the condition (D). Then, $t_{j,k}(x, y)$ depends upon j and k only in the combination of their difference $k - j$, and we can set

$$t_{j,k}(x, y) = d_{k-j}(x, y)$$

for some $d_i(x, y) \in \text{End}A$. Therefore, Eq.(4.17) becomes

$$[T_j(x, y), \rho_k(z)] = \rho_k(d_{j+k}(x, y)z) \tag{4.19}$$

which is Eq.(3.4). Here, we have changed $j \to 3 - j$.

Finally, since L is a Lie algebra, Eq.(3.4d) follows from Eqs.(3.4a) and (3.4c), satisfying all relations in Eqs.(3.4). As the result, Corollary 3.4 implies A to be a normal triality algebra with the validity of Eq.(4.9).

In order to show that $\tilde{L}$ is a ideal of L, we first note

$$[\rho_j(A), t] \subseteq \rho_j(A) \tag{4.20}$$

by Lemma 4.1. Moreover, we calcultate

$$[T_{3-j}(x,y), t] = [[\rho_j(x), \rho_j(y)], t]$$
$$= -[[\rho_j(y), t], \rho_j(x)] - [[t, \rho_j(x)], \rho_j(y)] \subseteq [\rho_j(A), \rho_j(A)] \subseteq T_{3-j}(A, A)$$

so that we have

$$[T_j(A, A), t] \subseteq T_j(A, A). \tag{4.21}$$

If L is invariant under S_4, we define $\overline{x}$ for any $x \in A$ by

$$\tau\rho_0(x) = -\rho_0(\overline{x}), \ (\text{i.e.}\tau x = -\bar{x}) \tag{4.22}$$

for the transposition $\tau = (1, 2) \in S_4$. We can then prove that $x \to \overline{x}$ is a involution of A.□

Remark 4.5

If L is simple, and if $\tilde{L}$ is *not* trivial, then $\tilde{L} = L$. Suppose that L is *not* simple, and S_4-invariant. Then, both t and $T(A, A)$ is S_3-invariant since $\phi\tau_\mu\phi^{-1} = \tau_{\mu+1}$, so that $L/\tilde{L} = t/T(A, A)$ is now S_3-invariant. A coordinatization of any Lie algebra which is invariant under S_3, or more generally *di*-cyclic group has been given in [E-O.09], [E-O.11].

Remark 4.6

It has been noted in [E-O.08] that any simple Lie algebra over the algebraicaly closed field of characteristic zero is S_4-invariant so that all these algebras can be constructed by some normal triality algebras. We will study some cases below.

Example 4.7(a)

The $so(3)$ Lie algebra defined by

$$[e_i, e_j] = \sum_{k=1}^{3} \varepsilon_{ijk} e_k, \ (i, j = 1, 2, 3)$$

is S_4-invariant. First, $\phi = (1,2,3)$ and $\tau = (1,2)$ are given by

$$\phi : e_1 \to e_2 \to e_3 \to e_1, \; \tau : e_1 \leftrightarrow e_2, \; e_3 \to -e_3$$

while the Klein's 4-group K_4 acts as

$$\begin{aligned} \tau_1 &: e_1 \to e_1, \; e_2 \to -e_2, \; e_3 \to -e_3 \\ \tau_2 &: e_2 \to e_2, \; e_1 \to -e_1, \; e_3 \to -e_3 \\ \tau_3 &: e_3 \to e_3, \; e_1 \to -e_1, \; e_2 \to -e_2. \end{aligned}$$

Then, $g_i = Fe_i$ with $T_j(A,A) = 0$. Hence, the resulting normal triality algebra or structurable algebra A is isomorphic to the field F itself. Similary, we note that the quaternion algebra is also S_4-invariant.

Example 4.7(b) (*$so(N)$* **algebra for** $N \geq 4$)

The $so(N)$ Lie algebra is defined by $J_{\mu\nu} = -J_{\nu\mu}$ satisfying

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\mu\alpha} J_{\nu\beta} - \delta_{\nu\alpha} J_{\mu\beta} - \delta_{\mu\beta} J_{\nu\alpha} + \delta_{\nu\beta} J_{\mu\alpha}$$

for $\mu, \nu, \alpha, \beta = 1, 2, \cdots, N$. It is clearly invariant under the symmetric group S_N permuting N indices $1, 2, \cdots, N$. For $N \geq 4$, it is then invariant also under its sub-group S_4 which permutes 4 indices $1, 2, 3$ and 4, but leaves other indices $5, 6, \cdots, N$ being unchanged. Then, the decomposition Eq.(4.7) of $L = so(N)$ by the Klein's 4-group is readily computed to yield

$$\begin{aligned} t &: \begin{cases} (1)\ J_{1j} + J_{2j} + J_{3j} + J_{4j} \ (j \geq 5) \\ (2)\ J_{ij}, \ (i, j \geq 5) \end{cases} \\ g_3 &: \begin{cases} (1)\ J_{13} + J_{24} \\ (2)\ J_{14} + J_{23} \\ (3)\ J_{1j} + J_{2j} - J_{3j} - J_{4j}, \ (j \geq 5) \end{cases} \\ g_1 &: \begin{cases} (1)\ J_{12} - J_{34} \\ (2)\ J_{13} - J_{24} \\ (3)\ J_{1j} - J_{2j} - J_{3j} + J_{4j}, \ (j \geq 5) \end{cases} \\ g_2 &: \begin{cases} (1)\ J_{12} + J_{34} \\ (2)\ J_{14} - J_{23} \\ (3)\ J_{1j} - J_{2j} + J_{3j} - J_{4j}. \ (j \geq 5) \end{cases} \end{aligned}$$

Assuming that the field F is of charachteristic $\neq 2$, we set

$$\begin{aligned} e &= \tfrac{1}{2}(J_{13} + J_{24} - J_{14} - J_{23}), \\ f_0 &= \tfrac{1}{2}(J_{13} + J_{24} + J_{14} + J_{23}), \\ f_{j-4} &= \tfrac{1}{4}(J_{1j} + J_{2j} - J_{3j} - J_{4j}), \ (j \geq 5). \end{aligned}$$

Then, by Eq,(4.13), we calculate

$$A = g_3 = \text{span} < e, f_\mu, \ (\mu = 0, 1, 2, \cdots, N-4) > \tag{4.23}$$

to be a unital commutative algebra with the multiptication table of

$$ef_\mu = f_\mu e = f_\mu, \ f_\mu f_\nu = \delta_{\mu\nu} e \tag{4.24}$$

for $\mu, \nu = 0, 1, 2, \cdots, N$. If we further introduce a symmetric bi-linear non-degenerate form $< \cdot | \cdot >$ in A by

$$< f_\mu | f_\nu > = \delta_{\mu\nu}, \ < f_\mu | e > = < e | f_\mu > = 0, \ < e | e > = 1,$$

A is a quadratic algebra satisfying

$$x^2 - 2 < x | e > x + < x | x > e = 0 \tag{4.25}$$

for any $x \in A$. Especially, A is Jordan algebra. Therefore, A is also a structurable algebra with $\overline{x} = x$. Note that Eq.(4.22) will give, contrarily $\overline{e} = e$ but $\overline{f}_\mu = -f_\mu$, for $\mu = 0, 1, \cdots, N-4$.

In this case, we have $L(A) = so(N)$, $L_j(A) = so(N-2) \oplus gl(1)$ and $T_j(A, A) = so(N-3)$ as well as $T_1(x, y) = T_2(x, y) = T_0(x, y)$ in accordance with $d_1(x, y) = d_2(x, y) = d_0(x, y)$ (see discussion given after Eq.(3.11)). We also note that $L_j(A)$ is still S_{N-4}-invariant, permuting indices $5, 6, -, N$.

Remark 4.8(Some Lie superalgebras)

Some Lie superalgebras are also S_4-invariant, and we can apply the same tecnique to show the triality (see Remark 1.9). Consider for example of the Lie superalgebra $osp(N, 2)$ for $N \geq 4$. They can be invariant under the S_4 symmetry by extending the action of S_4 of its even-part $L_{\overline{0}} = so(N)$ $(N \geq 4)$. Then the resulting normal triality super-algebra A has its even part $A_{\overline{0}}$ given by Eq.(4.23), while its odd part $A_{\overline{1}}$ is

$$A_{\overline{1}} = \text{span} < \xi_1, \xi_2 >$$

satisfying

$$\xi_1\xi_2 = -\xi_2\xi_1 = e, \quad \xi_1\xi_1 = \xi_2\xi_2 = 0, \ e\xi_\alpha = \xi_\alpha e = \xi_\alpha \ (\alpha = 1, 2)$$
$$f_\mu\xi_\alpha = \xi_\alpha f_\mu = 0, \quad (\mu = 0, 1, 2, \cdots, N-4, \text{and } \alpha = 1, 2,).$$

The cases of Lie superalgebras $G(3)$ and $F(4)$ have been disscussed in [E-O,08].

Example 4.9($sl(N)$ Lie algebra for $N \geq 4$)

The $sl(N)$ Lie algebra is specified by the commutation relation

$$[X^\mu_\nu, X^\alpha_\beta] = \delta^\mu_\beta X^\alpha_\nu - \delta^\alpha_\nu X^\mu_\beta, \; \sum_{\mu=1}^{N} X^\mu_\mu = 0$$

for $\mu, \nu, \alpha, \beta = 1, 2, \cdots, N$, which is invariant under S_N again. Assuming $N \geq 4$, and restricting ourselves to its sub-group S_4 as in Example 4.7b, we find the resulting normal triality algebra to be given by

$$A = g_3 = \text{span} < e, f, x^\mu, x_\mu, \; \mu = 0, 1, 2, \cdots, N-4 > \tag{4.26}$$

where we have set

$$\begin{aligned} e &= \tfrac{1}{2}\{(X^3_1 - X^1_3 + X^4_2 - X^2_4) + (X^2_3 - X^3_2 + X^1_4 - X^4_1)\} \\ f &= \tfrac{1}{2}\{(X^1_1 + X^2_2 - X^3_3 - X^4_4) - (X^1_2 + X^2_1 - X^3_4 - X^4_3)\} \\ x_{j-4} &= X^1_j + X^2_j - X^3_j - X^4_j, \; (j \geq 5) \\ x^{j-4} &= X^j_1 + X^j_2 - X^j_3 - X^j_4 \; (j \geq 5) \end{aligned}$$

while x_0 and x^0 are defined by

$$\begin{aligned} x_0 &= g - k, \text{ and } x^0 = g + k, \\ g &= \tfrac{1}{2}\{(X^1_1 + X^2_2 - X^3_3 - X^4_4) + (X^1_2 + X^2_1 - X^3_4 - X^4_3)\} \\ k &= \tfrac{1}{2}\{(X^3_1 - X^1_3 + X^4_2 - X^2_4) - (X^2_3 - X^3_2 + X^1_4 - X^4_1)\}. \end{aligned}$$

They satisfy by Eq.(4.13)the multiplication table of

(1) $ef = fe = -f$, but $ex = xe = x$ for $x = x_\mu$ and x^μ,

(2) $ff = e$,

(3) $fx_\mu = -x_\mu f = x_\mu, \; fx^\mu = -x^\mu f = -x^\mu$,

(4) $x_\mu x_\nu = 0 = x^\mu x^\nu$,

(5) $x_\mu x^v = 2\delta^\nu_\mu(f - e), \; x^\nu x_\mu = -2\delta^\nu_\mu(f + e)$

for $\mu, \nu = 0, 1, 2, \cdots, N-4$, reproducing the result of Example 2.5.

For $L(A) = sl(N)$, we have $L_i(A) = sl(N-2) \oplus gl(1)$, and $T_j(A, A) = sl(N-3)$.

Also in the Example (2.5), we have seen that this algebra has more than one involution and the involution 1 corresponds to the case of A^* being structurable. Note that the involution 3 is the one obtained by Eq.(4.22), i.e, $\tau\rho_0(x) = -\rho_0(\overline{x})$.

Example 4.10(Lie algebra E_8)

The exceptional Lie algebra E_8 as well as Lie superalgebras $G(3)$ and $F(4)$ are S_4-invariant, and are discussed in [E-O.08]. We will not go into details.

Remark 4.11(Tetrahedron Algebra)

The tetrahedron Lie algebra ⊠ of Hartwig and Terlliger [H-T,07] is generated by

$$\{X_{ij}, |i, j \in I,\ i \neq j\}, I = \{0, 1, 2, 3\}$$

with

(1) For distinct $i, j \in I,\ X_{ij} + X_{ji} = 0$

(2) For mutually district $k, i, j \in I$,

$$[X_{ki}, X_{ij}] = 2(X_{ki} + X_{ij})$$

(3) For mutually district $h, i, j, k \in I$,

$$[X_{ki}, [X_{ki}, [X_{ki}, X_{jh}]]] = 4[X_{ki}, X_{jh}].$$

It is clearly S_4-invariant, and we have the decomposition

$$\boxtimes = \Omega \oplus \Omega' \oplus \Omega'',$$

where Ω $(resp\, \Omega')$ and $(resp\, \Omega'')$ is a sub-algebra of ⊠ generated by X_{12}, X_{03} (resp. X_{23}, X_{01}) and (resp.X_{31}, X_{02}). All Ω, Ω' and Ω'' are Onsager Lie algebras.

A remarkable fact is that it is isomorphic to three point $sl(2)$ loop algebra by

$$\Phi : \boxtimes \to sl(2) \otimes F(t, \frac{1}{t}, \frac{1}{1-t})$$

for a indefinite variable t. For details, see [H-T.07] and [E.07]. However, the relationship between these facts and Theorem 4.4 is not transparent.

In ending this section, let us consider a further study of the structure of the normal triality algebra in Theorem 4.4. Let

$$k(X,Y) := \mathrm{Tr}(\mathrm{ad}\, X \;\mathrm{ad}\, Y) \tag{4.27}$$

for $X, Y \in \tilde{L}(A)$ be the Killing form of $\tilde{L}(A)$, where "ad" implies the adjoint representation. We will then prove the following Theorem:

Theorem 4.12

Let A be the normal triality algebra associated with the finite-dimensional A_4-invariant Lie algebra $\tilde{L}(A)$ as in Theorem 4.4. Then, there exists a bilinear form $< \cdot|\cdot >$ in A, satisfying

(1)

$$< y|x > = < x|y > \tag{4.28a}$$

(2)

$$< xy|z > = < x|yz > \tag{4.28b}$$

(3)

$$\begin{aligned} < x|d_j(z,w)y > &= - < d_j(z,w)x|y > = < z|d_{3-j}(x,y)w > \\ &= - < d_{3-j}(x,y)z|w > . \end{aligned} \tag{4.28c}$$

Moreover, if A is involutive with involution map $x \to \bar{x}$ in addition, we have

$$< \bar{x}|\bar{y} > = < x|y > . \tag{4.29}$$

Further, the Killing form $k(X,Y)$ of $\tilde{L}(A)$ are given by

(1)

$$k(\rho_i(x), \rho_j(y)) = \delta_{ij} < x|y > \tag{4.30a}$$

(2)

$$k(\rho_i(x), T_j(y,z)) = 0 \tag{4.30b}$$

(3)

$$\begin{aligned} k(T_i(x,y), T_j(z,w)) &= - < x|d_{j-i}(z,w)y > = < d_{j-i}(z,w)x|y > \\ &= - < z|d_{i-j}(x,y)w > = < d_{i-j}(x,y)z|w > \end{aligned} \tag{4.30c}$$

for any $x, y, z, w \in A$ and for any $i, j = 0, 1, 2$.

In order to prove this Theorem, we start with the following observation. Let $e_1, e_2, \cdots, e_N$ with $N = Dim\ \tilde{L}(A)$ be a basis of $\tilde{L}(A)$ with the multiplication table of

$$[e_\mu, e_\nu] = \sum_{\lambda=1}^{N} C^\lambda_{\mu\nu} e_\lambda, \quad (\mu, \nu, \lambda = 1, 2, \cdots, N)$$

for structure constants $C^\lambda_{\mu\nu} \in F$ of $\tilde{L}(A)$. Then, we see that

$$\mathrm{Tr}(\mathrm{ad}\ e_\mu\ \mathrm{ad}\ e_\nu)$$

is completely determined in terms of the structure constants $C^\lambda_{\mu\nu}$. For any $\sigma \in \mathrm{Auto}(\tilde{L}(A))$,

$$e'_\mu = \sigma(e_\mu)$$

will offer another basis of $\tilde{L}(A)$ with the same multiplication table of

$$[e'_\mu, e'_\nu] = \sum_{\lambda=1}^{N} C^\lambda_{\mu\nu}\ e'_\lambda$$

with the same structure constants $C^\lambda_{\mu\nu}$. Since the trace operation is independent of the choice of the basis, these imply the validity of

$$\mathrm{Tr}(\mathrm{ad}\ \sigma(e_\mu)\mathrm{ad}\ \sigma(e_\nu)) = \mathrm{Tr}(\mathrm{ad}\ e_\mu\ \mathrm{ad}\ e_\nu),$$

so that $k(\sigma(X), \sigma(Y)) = k(X, Y)$. In other words, the Killing form is invariant under action of the automorphisum group Auto $(\tilde{L}(A))$.

If we choose $\sigma = \phi = (1, 2, 3) \in Z_3$, then we find

$$k(\phi_1(x), \phi_1(y)) = k(\phi_2(x), \phi_2(y)) = k(\phi_3(x), \phi_3(y)),$$

where $\phi_i (i = 0, 1, 2)$ are denoted by $\phi_1(x) = \phi(x) = \rho_1(x), \phi_2(x) = \phi^2(x) = \rho_2(x)$ and $\phi_3(x) = \phi^3(x) = x = \rho_3(x)$

Moreover, if we consider the choice of $\sigma = \tau_j$, $(j = 1, 2, 3)$ being the Klein's four group. it gives

$$k(\phi_i(x), \phi_j(y)) = 0, \text{ if } i \neq j.$$

Therefore, we can write

$$k(\phi_i(x), \phi_j(y)) = \delta_{ij} < x|y > \tag{4.30c}$$

for some bi-linear form $< x|y >$ as in Eq.(4.30a). Then, Eq.(4.28a), i.e. $< y|x > = < x|y >$ follows immediately from the fact that the left side of Eq.(4.30a) is symmetric in $x \leftrightarrow y$ and $i \leftrightarrow j$.

We next consider the trace identity of

$$\begin{aligned}
&\mathrm{Tr}(\mathrm{ad}\ \rho_1(x)[\mathrm{ad}\ \rho_2(y), \mathrm{ad}\ \rho_3(z))]) \\
&= \mathrm{Tr}([\mathrm{ad}\ \rho_1(x), \mathrm{ad}\ \rho_2(y)]\mathrm{ad}\ \rho_3(z))
\end{aligned}$$

which gives

$$\mathrm{Tr}(\mathrm{ad}\ \rho_1(x)\ \mathrm{ad}\ \rho_1(yz)) = \mathrm{Tr}(\mathrm{ad}\ \rho_3(xy)\ \mathrm{ad}\ \rho_3(z)),$$

i.e. $< x|yz > = < xy|z >$.

Finally, we calculate

$$\begin{aligned}
k(T_i(x,y), T_j(z,w)) &= \mathrm{Tr}(\mathrm{ad}\ T_i(x,y)\ \mathrm{ad}\ T_j(z,w)) \\
&= \mathrm{Tr}([\mathrm{ad}\ \rho_{3-i}(x), \mathrm{ad}\ \rho_{3-i}(y)]\mathrm{ad}\ T_j(z,w)) \\
&= \mathrm{Tr}(\mathrm{ad}\ \rho_{3-i}(x)[\mathrm{ad}\ \rho_{3-i}(y), \mathrm{ad}\ T_j(z,w)]) \\
&= -\mathrm{Tr}(\mathrm{ad}\ \rho_{3-i}(x)\ \mathrm{ad}\ \rho_{3-i}(d_{j-i}(z,w)y)) \\
&= - < x|d_{j-i}(z,w)y > .
\end{aligned} \tag{4.31}$$

Moreover, since the left side of Eq.(4.31) is anti-symmetric in $x \leftrightarrow y$, but symmetric for $x \leftrightarrow z, y \leftrightarrow w$ and $i \leftrightarrow j$, it also yields Eqs.(4.28c) and (4.30c).

Last, suppose that A is involutive with the involution map $x \to \bar{x}$. Then, $\sigma = (1,2)$ as is given by Eq.(3.17) is also an automorphysm of $\tilde{L}(A)$ so that it yields $< \bar{x}|\bar{y} > = < x|y >$. These complete the proof of Theorem 4.12 □

Proposition 4.13

If the bi-linear form $< \cdot|\cdot >$ for A is degenerate, then so is the Killing form $k(X,Y)$ of the Lie algebra $\tilde{L}(A)$. Especially, if $k(X,Y)$ is non-degenerate, then $< \cdot|\cdot >$ is non-degenerate. Conversly, if $< \cdot|\cdot >$ is non-degenerate, $k(X,Y)$ is non-degenerate, provided that $\tilde{L}(A)$ does not contain any center element.

Proof

Suppose that $<\cdot|\cdot>$ is degenerate. Then, there exists $x_0 \in A$ satisfying $< x_0|x >= 0$ for any $x \in A$, and we set $X_0 = \rho_i(x_0)$ for some $i = 0, 1, 2$. We then calculate

$$k(X_0, \rho_j(x)) = k(\rho_i(x_0), \rho_j(x)) = \delta_{ij} < x_0|x >= 0$$

and

$$k(X_0, T_j(x, y)) = k(\rho_i(x_0), T_j(x, y)) = 0$$

so that $k(X_0, X) = 0$ for any $X \in \tilde{L}(A)$. Therefore, $k(X, Y)$ is degenerate.

Conversely, suppose that $<\cdot|\cdot>$ is non-degenerate, and there exists $X_0 \in \tilde{L}(A)$ satisfying $k(X_0, X) = 0$ for any $X \in \tilde{L}(A)$. Writing

$$X_0 = \sum_{i=1}^{3}\sum_{\lambda=1}^{N} C_{i\lambda}\rho_i(e_\lambda) \oplus \sum_{i=1}^{3}\sum_{\mu,\nu=1}^{N} C_{i,\mu\nu}T_i(e_\mu, e_\nu), \tag{4,32}$$

for some constant $C_{i\lambda}, C_{i,\mu\nu} \in F$, we calculate

$$0 = k(X_0, \rho_j(x)) = \sum_{\lambda=1}^{N} C_{j\lambda} < e_\lambda|x >,$$

$$0 = k(X_0, T_j(x, y)) = -\sum_{\lambda=1}^{3}\sum_{\mu,\nu=1}^{N} C_{i,\mu\nu} < x|d_{i-j}(e_\mu, e_\nu)y >$$

which yields

$$\sum_{\lambda=1}^{N} C_{j\lambda}e_\lambda = 0 \tag{4.33a}$$

and

$$\sum_{i=1}^{3}\sum_{\mu,\nu=1}^{N} C_{i,\mu\nu}d_{i-j}(e_\mu, e_\nu) = 0 \tag{4.33b}$$

for any $j = 0, 1, 2$ in view of the non-degeneracy of $<\cdot|\cdot>$. Especially, Eqs.(4.32) and (4.33a) imply

$$X_0 = \sum_{i=1}^{3}\sum_{\mu,\nu=1}^{N} C_{i,\mu\nu}T_i(e_\mu, e_\nu). \tag{4.34}$$

We compute then

$$[X_0, \rho_j(x)] = \sum_{i=1}^{3} \sum_{\mu,\nu=1}^{N} C_{i,\mu\nu} \rho_j(d_{i+j}(e_\mu, e_\nu)x) = 0$$

by Eq.(4.33b) for $j \to -j$, while we note

$$[X_0, T_j(x,y)] = \sum_{i=1}^{3} \sum_{\mu,\nu=1}^{N} C_{i,\mu\nu} \{T_j(d_{i-j}(e_\mu, e_\nu)x, y) + T_j(x, d_{i-j}(e_\mu, e_\nu)y)\}$$

$$= 0$$

again by Eq.(4.33b). Therefore, we have

$$[X_0, \tilde{L}(A)] = 0$$

and X_0 is a center element of $\tilde{L}(A)$. Especially, if $\tilde{L}(A)$ does not contain any center, then $X_0 = 0$ and $k(X,Y)$ is non-degenerate. □

Hereafter in this section, we will assume the non-degeneracy of $< \cdot | \cdot >$, and recall the following Theorem of Dieudonne ([S.66]):

Suppose that a finite dimensional algebra A has a non-degenerate associative bi-linear form $< \cdot | \cdot >$ (*that is*, $< xy|z >=< x|yz >$). If A possesses no ideal B satisfying $BB = 0$, then A is a direct sum of simple ideals. We can then show.

Proposition 4.14

Let the algebra A be to have a ideal B satisfying $BB = 0$. Moreover, assume the validity of both conditions (B) and (C) in section one. Then, the associated Lie algebra $\tilde{L}(A)$ contains a solvable ideal.

For a proof of this Proposition, we set

$$B_0 = AB + BA. \tag{4.36}$$

Then, $0 \neq B_0 \subseteq B$ and B_0 is a non-zero ideal of A satisfying $< B_0|B >= 0$, since we calculate $(AB)A \subseteq BA \subseteq B_0$, as well as $< AB|B >=< A|BB >= 0$ etc.. We then note

Lemma 4.15

Under the assumpsion as in Proposision 4.14, we have

(i)

$$d_j(A,A)B_0 \subseteq B_0, \quad d_j(A,B)A \subseteq B_0 \tag{4.37a}$$

(ii)

$$d_j(A,B)B_0 = 0, \quad d_j(B_0,B_0) = 0 \tag{4.37b}$$

for any $j = 0, 1, 2$.

Proof

If we note $d_1(x,y) = R(y)L(x) - R(x)L(y)$, we calculate

$$d_1(x,B)y = (xy)B - (By)x \subseteq B_0$$

$$d_1(x,y)B = (xB)y - (yB)x \subseteq B_0$$

for any $x, y \in A$, since B is a ideal of A. Moreover

$$d_1(x,B)B = (xB)B - (BB)x \subseteq BB = 0$$

$$d_1(B,B)x = (Bx)B - (Bx)B \subseteq BB = 0$$

so that Eq.(4.37) hold for $j = 1$. Similarly, for $j = 2$, we note

$$d_2(x,y) = L(y)R(x) - L(x)R(y)$$

and we can verify the validity of Eq.(4.37) in the same way.

We can then verify the validity of Eqs.(4.37) for $j = 0$ as follows

$$d_0(A,A)(AB) = (d_1(A,A)A)B + A(d_2(A,A)B) \subseteq AB \subseteq B_0,$$

$$d_0(A,A)(BA) = (d_1(A,A)B)A + B(d_2(A,A)B) \subseteq BA \subseteq B_0,$$

so that

$$d_0(A,A)B_0 \subseteq B_0.$$

Also, we note

$$d_0(A,B)(AA) = (d_1(A,B)A)A + A(d_2(A,B)A) \subseteq BA + AB = B_0.$$

But by the condition (B), we have $AA = A$, which gives

$$d_0(A,B)A \subseteq B_0.$$

We can similarly check the validity of Eq.(4.37b) for $j = 0$. □

Now, we can prove Proposition 4.14 as follows.

Set

$$L^0 = \text{span} < \rho_j(B_0),\ T_j(A, B_0),\ j = 0, 1, 2 > .$$

Then, by the Lemma 4.15, L^0 is a proper ideal of $\tilde{L}(A)$.

Moreover, we calculate

$$L^{(1)} := [L^0, L^0] = \text{span} < T_j(B_0, B_0),\ j = 0, 1, 2 >$$

and

$$L^{(2)} := [L^{(1)}, L^{(1)}] = 0.$$

As the consequense, L^0 is solvable, and hence $\tilde{L}(A)$ contains a solvable ideal.

Last in ending this section, we simply note

Remark 4.16(see Proposition 3.4 of [O.05])

Let A be an algebra with a bi-linear non-degenerate form $< \cdot | \cdot >$ satisfying Eq.(4.28) for some $d_j(\cdot, \cdot) \in \text{End } A$. However, explicit forms for $d_j(x, y)'s$ are not specified at all. Then, any one of the following four statements implies the validity all others.

(1) We have the triality relation:

$$d_j(u, v)(xy) = (d_{j+1}(u, v)x)y + x(d_{j+2}(u, v)y)$$

for all values of $j = 0, 1, 2$, with $d_{j+3}(u, v) = d_j(u, v)$.

(2) The triality relations holds only for one value of j, say $j = 0$ for example,

$$d_0(u, v)(xy) = (d_1(u, v)x)y + x(d_2(u, v)y).$$

(3) $< d_0(u, v)z|xy > + < d_1(u, v)x|yz > + < d_2(u, v)y|zx >= 0$,

(4) $d_0(x, yz) + d_1(z, xy) + d_2(y, zx) = 0$.

Moreover, if A is involutive with the involution map $x \to \bar{x}$ and if we have $< \bar{x}|\bar{y} >=< x|y >$, then Eq.(4.28b) becomes

$$< \bar{x}|y \star z >=< \bar{y}|z \star x >=< \bar{z}|x \star y >$$

for the conjugate algebra A^* of A.

5. Pre-structurable Algebra

Although we have already defined a pre-structurable algebra by Def.1.6, we will here introduce the following slightly more generalization for a later purpose.

Def.5.1

Let A^* be a involutive algebra with the bi-linear product $x \star y$ and the involution $x \to \overline{x}$. Suppose that it satisfies the triality relation

$$\overline{d_j(x,y)}(u \star v) = (d_{j+1}(x,y)u) \star v + u \star (d_{j+2}(x,y)v) \tag{5.1}$$

for $d_j(x,y) \in \text{End}\, A^*$ $(j = 0,1,2)$ given by

$$d_1(x,y) = l(\overline{y})l(x) - l(\overline{x})l(y), \tag{5.2a}$$

$$d_2(x,y) = r(\overline{y})r(x) - r(\overline{x})r(y) \tag{5.2b}$$

$$\begin{aligned} d_0(x,y) &= r(\overline{x} \star y - \overline{y} \star x) + l(y)l(\overline{x}) - l(x)l(\overline{y}) \\ &= l(y \star \overline{x} - x \star \overline{y}) + r(y)r(\overline{x}) - r(x)r(\overline{y}). \end{aligned} \tag{5.3}$$

We call then A^* be an almost pre-structurable algebra. Note that if A^* is unital in addition, then A^* is pre-sturcturable. Moreover, setting

$$Q(x,y,z) = d_0(x, \overline{y \star z}) + d_1(z, \overline{x \star y}) + d_2(y, \overline{z \star x}) \tag{5.4}$$

as before, we call a pre-sturcturable algebra be structurable when we have $Q(x,y,z) = 0$ furthermore.

Def.5.2

Let A^* be a involutive algebra. We introduce multiplication operators by (see [A-F,93])

$$A(x,y,z)w := \{(w \star x) \star \overline{y}\} \star z - w \star \{x \star (\overline{y} \star z)\} \tag{5.5a}$$

$$B(x,y,z)w := \{(w \star x) \star \overline{y}\} \star z - w \star \{(x \star \overline{y}) \star z\} \tag{5.5b}$$

$$C(x,y,z)w := \{x \star (\overline{y} \star \overline{w})\} \star z - (x \star \overline{y}) \star (\overline{w} \star z) \tag{5.5c}$$

$$C'(x,y,z)w = C(x,y,z)\overline{w} = \{x \star (\overline{y} \star w)\} \star z - (x \star \overline{y}) \star (w \star z) \tag{5.5d}$$

for $x,y,z,w \in A^*$.

Lemma 5.3

If A^* is an almost pre-structurable algebra, we have
(1)

$$A(x,y,z) - A(y,x,z) = A(z,x,y) - A(z,y,x) \tag{5.6a}$$

and
(2)

$$B(z,x,y) - B(z,y,x) = C^{'}(y,x,z) - C^{'}(x,y,z). \tag{5.6b}$$

Conversly, if an involutive algebra A^* with $d_j(x,y)$'s being given by Eqs.(5.2) and (5.3) satisfies Eq.(5.6a) or (5.6b), respectively, then the triality relation Eq.(5.1) holds respectively for $j = 1$ and 2 or for $j = 0$.

Proof

We may easily verify that
(1) Eq.(5.1) for $j = 1$ with $d_0(x,y) = r(\overline{x} \star y - \overline{y} \star x) + l(y)l(\overline{x}) - l(x)l(\overline{y})$ is rewritten as

$$\{A(w,x,y) - A(w,y,x)\}z = \{A(x,y,w) - A(y,x,w)\}z.$$

Similarly, Eq.(5.1) for $j = 2$ with $d_0(x,y) = l(y\star\overline{x} - x\star\overline{y}) + r(y)r(\overline{x}) - r(x)r(\overline{y})$ is equivalent to the same relation, if we take the involution of the relation.
(2) Eq.(5.1) for $j = 0$ is similarly shown to be equivalent to the validity of Eq.(5.6b). □

Proposition 5.4(see [A-F,93])

Let A^* be now a pre-structurable algebra. We then also have

(1)

$$B(x,y,z) - B(y,z,x) = B(z,x,y) - B(z,y,x), \tag{B}$$

(2)

$$[x - \overline{x}, y, z] = -[y, x - \overline{x}, z] = [y, z, x - \overline{x}], \tag{sk}$$

(3)

$$[x, \overline{y}, z] - [y, \overline{x}, z] = [z, \overline{x}, y] - [z, \overline{y}, x] = [z, x, \overline{y}] - [z, y, \overline{x}] \tag{A.1}$$

where $[x, y, z]$ is the associater of A^* defined by

$$[x, y, z] = (x \star y) \star z - x \star (y \star z). \tag{5.7}$$

Proof

If we note $A(x, y, z)e = [x, \overline{y}, z]$ for the unit element e of A^*, then Eq.(5.6a) becomes

$$[x, \overline{y}, z] - [y, \overline{x}, z] = [z, \overline{x}, y] - [z, \overline{y}, x] \tag{5.6$'$}$$

which is a part of Eq.(A1) consistent with Eq.(5.3). Also from Eq.(5.5), we see

$$A(x, y, z)w = B(x, y, z)w - w \star [x, \overline{y}, z] \tag{5.7$'$}$$

so that Eq.(5.6a) together with Eq.(5.6)′ and (5.7)′ gives Eq.(B).

If we next set $y = e$ in Eq.(B), it yields

$$[w, x - \overline{x}, z] = [w, z, \overline{x} - x].$$

Taking the involution of this relation, and changing the notation suitably we obtain Eq.(sk). Other relations can be similarly proved. □

Lemma 5.5

Let A^* be a pre-structurable algebra. Then

$$D(x, y) = d_0(x, y) + d_1(x, y) + d_2(x, y) \tag{5.8a}$$

is a derivation of A^*, satisfying

$$\overline{D(x, y)} = D(\overline{x}, \overline{y}) = D(x, y). \tag{5.8b}$$

Proof

Form Eqs.(5.2), we see that $d_j(x, y)$'s satisfy

$$\overline{d_j(x, y)} = d_{3-j}(\overline{x}, \overline{y}) \tag{5.9a}$$

and

$$d_0(x, y)z + d_0(y, z)x + d_0(z, x)y = 0. \tag{5.9b}$$

Then, $\overline{D(x, y)} = D(\overline{x}, \overline{y})$ immediately follows from Eq.(5.8a) and (5.9a).

We note

$$\begin{aligned} D(x,y)z &= z\star(\overline{x}\star y-\overline{y}\star x)+y\star(\overline{x}\star z)-z\star(\overline{y}\star x) \\ &+\overline{y}\star(z\star x)-\overline{x}\star(y\star z)+(z\star x)\star\overline{y}-(z\star y)\star\overline{x}. \end{aligned}$$

by Eqs.(5.2). We then calculate

$$\{D(x,y)-D(\overline{x},\overline{y})\}z=[z,x,\overline{y}]-[z,y,\overline{x}]+[z,\overline{y},x]-[z,\overline{x},y]=0$$

by Eq.(A1) so that we have $D(x,y)=D(\overline{x},\overline{y})$. Finally, summing over $j=0,1,2$ in Eq.(5.1), we obtain

$$\overline{D(x,y)}(u\star v)=(D(x,y)u)\star v)+u\star(D(x,y)v)$$

which shows $D(x,y)$ to be a derivation of A^* in view of Eq.(5.8b). □

We next consider two sets of

$$S=\{x|\overline{x}=x,\ x\in A^*\}$$

$$H=\{x|\overline{x}=-x,\ x\in A^*\}. \tag{5.10}$$

Then, if the underlying field F is of characteristic $\neq 2$, Eq.(sk) indicates that H is a generalized alternative nucleus of A^*. As the consequence,H is a Malcev algebra with respect to the commutor product $[x,y]^*=x\star y-y\star x$(see [P-S.04]).

Theorem 5.6 ([K-O.14])

Let A^* be a pre-structurable algebra. We then have

(1) $Q(x,y,z)w$ is totally symmetric in $x,y,z,w\in A^*$.

(2) $Q(x,y,z)w=0$ identically, if at least one of x,y,z and w is the unit element e of A^*.

(3) Supose that the underlying field F is of charachteristic $\neq 2$. Then, $Q(x,y,z)w=0$ identically again, provided that at least one of x,y,z, and w is an element of H.

(4) $\overline{Q(x,y,z)}=Q(\overline{x},\overline{y},\overline{z})=Q(x,y,z)$ is a derivation of A^*.

(5) $3Q(x,y,z)=D(x,\overline{y}\star\overline{z})+D(y,\overline{z}\star\overline{x})+D(z,\overline{x}\star\overline{y})$

(6) $[Q(x,y,z),Q(u,v,w)] = Q(Q(x,y,z)u,v,w)+Q(u,Q(x,y,z)v,w)+Q(u,v,Q(x,y,z)w)$.

For the proof of this Theorem, we start from the following Lemma.

Lemma 5.7

We have

$$Q(x,y,z)e = 0.$$

Proof

We calculate

$$Q(x,y,z) = r(\overline{x}\star(\overline{z}\star\overline{y})) - r((y\star z)\star x) + l(\overline{z}\star\overline{y})l(\overline{x}) - l(x)l(y\star z)$$

$$+l(x\star y)l(z) - l(\overline{z})l(\overline{y}\star\overline{x}) + r(z\star x)r(y) - r(\overline{y})r(\overline{x}\star\overline{z}), \qquad (5.11)$$

from Eq.(5.2) and (5.4), so that we obtain

$$\begin{aligned} Q(x,y,z)e &= \overline{x}\star(\overline{z}\star\overline{y}) - (y\star z)\star x + (\overline{z}\star\overline{y})\star\overline{x} - x\star(y\star z) \\ &\quad +(x\star y)\star z - \overline{z}\star(\overline{y}\star\overline{x}) + y\star(z\star x) - (\overline{x}\star\overline{z})\star\overline{y} \\ &= -[\overline{x},\overline{z},\overline{y}] - [y,z,\overline{x}] + [\overline{z},\overline{y},\overline{x}] + [x,y,z] = 0 \end{aligned}$$

by Eq.(A.1). □

Lemma 5.8

(1)

$$\overline{Q(x,y,z)} = Q(\overline{x},\overline{z},\overline{y}) \qquad (5.12a)$$

(2)

$$\overline{Q(x,y,z)}(u\star v) = \{Q(y,z,x)u\}\star v + u\star\{Q(z,x,y)v\}. \qquad (5.12b)$$

Proof

Eq.(5.12a) is nothing but Eq.(1.37), while we note

$$\begin{aligned} \overline{d_0(x,\overline{y\star z})}(u\star v) &= \{d_1(x,\overline{y\star z})u\}\star v + u\star\{d_2(x,\overline{y\star z})v\} \\ \overline{d_1(z,\overline{x\star y})}(u\star v) &= \{d_2(z,\overline{x\star y})u\}\star v + u\star\{d_0(z,\overline{x\star y})v\} \\ \overline{d_2(y,\overline{z\star x})}(u\star v) &= \{d_0(y,\overline{z\star x})u\}\star v + u\star\{d_1(y,\overline{z\star x})v\}. \end{aligned}$$

Adding all these relations, we obtain Eq.(5.12b). □

We next first set $u = e$ and $v = w$, and also $v = e$ and $u = w$ in Eq.(5.12b) to find

$$\overline{Q(x,y,z)}w = Q(z,x,y)w = Q(y,z,x)w$$

where we used $Q(x,y,z)e = 0$ by Lemma 5.7. This implies the validity of

$$\overline{Q(x,y,z)} = Q(z,x,y) = Q(y,z,x). \qquad (5.12c)$$

Especially, letting further $x \to y \to z \to x$, this leads to

$$Q(z,x,y) = Q(y,z,x) = Q(x,y,z) \qquad (5.12d)$$

to be cyclically invariant, and then

$$\overline{Q(x,y,z)} = Q(x,y,z) \qquad (5.12e)$$

by Eq.(5.12c) again.

Moreover, since A^* is unital, its conjugate algebra A satisfies $ex = xe = \overline{x}$ so that the conditions (B) and (C) of section 1 are automatically satisfied. Especially, A is a pre-normal triality algebra, so that Eq.(1.36) holds with

$$Q(x,y,z)w = Q(w,y,z)x,$$

by Proposition 1.5. We then calculate

$$\begin{aligned} Q(x,y,z)w &= Q(w,y,z)x = Q(y,z,w)x = Q(x,z,w)y \\ &= Q(w,x,z)y = Q(y,x,z)w \end{aligned}$$

which yields $Q(x,y,z) = Q(y,x,z)$. Together with Eq.(5.12d), these imply that $Q(x,y,z)w$ is totally symmetric in x, y, z and w. Then Lemma 5.7 shows that $Q(x,y,z)w = 0$ if at least one of x, y, z and w coincides with the unit element e. Moreover Eqs.(5.12a,b,c), and (5.12d) imply also $\overline{Q(x,y,z)} = Q(\overline{x},\overline{y},\overline{z}) = Q(x,y,z)$ to be a derivation of A^*.

In order to prove the statement (3) of Theorem 5.6, we write generic element of S and H as x_0 and x_1 respectively, so that $\overline{x_0} = x_0$ and $\overline{x_1} = -x_1$. Then, $Q(\overline{x},\overline{y},\overline{z}) = Q(x,y,z)$ yield immediately $Q(x_1,y_1,z_1) = 0 = Q(x_0,y_0,z_1)$, provided that the field F is of charachteristic $\neq 2$. Moreover, we note

$$Q(x_1,y_1,z_0)w_1 = Q(x_1,y_1,w_1)z_0 = 0,$$

and

$$Q(x_1, y_1, z_0)w_0 = Q(x_1, w_0, z_0)y_1 = 0$$

so that we have also $Q(x_1, y_1, z_0) = 0$. This proves the statement (3) of the Theorem.

Also, Eq.(1.35) together with Eq.(5.12d) yields immediately the relation of

$$3Q(x,y,z) = D(x, \overline{y \star z}) + D(y, \overline{z \star x}) + D(z, \overline{x \star y}).$$

Therefore, it remains only to prove the final statement (6). To show it, we note the following:

Lemma 5.9

Let D be a derivation of A^* satisfying $\overline{D} = D$, then we have

$$[D, Q(u,v,w)] = Q(Du, v, w) + Q(u, Dv, w) + Q(u, v, Dw). \tag{5.13}$$

Proof

Since D is a derivation of A^*, we have

$$D(u \star v) = (Du) \star v + u \star (Dv)$$

which is equivalent to the validity of

$$[D, l(u)] = l(Du), \quad [D, r(v)] = r(Dv).$$

Moreover, $\overline{D} = D$ implies $\overline{Dx} = D\overline{x}$. Then, these are sufficient to prove Eq.(5.13). □

Since $D = Q(x,y,z)$ satisfies the condition of Lemma 5.9, these give

$$[Q(x,y,z), Q(u,v,w)] =$$

$$Q(Q(x,y,z)u, v, w) + Q(u, Q(x,y,z)v, w) + Q(u, v, Q(x,y,z)w).$$

These results complete the proof of Theorem 5.6.

Remark 5.10

If we choose $D = D(x,y)$ in Lemma 5.9, we have

$$[D(x,y), Q(u,v,w)] =$$

$$Q(D(x,y)u,v,w)+Q(u,D(x,y)v,w)+Q(u,v,D(x,y)w). \qquad (5.14a)$$

Moreover, we have also

$$[Q(u,v,w),D(x,y)]=D(Q(u,v,w)x,y)+D(x,Q(u,v,w)y) \qquad (5.14b)$$

by the following reason. Since A is a pre-normal triality algebra, we have Eq.(1.20),i.e.

$$[d_j(u,v),d_k(x,y)]=d_k(d_{j-k}(u,v)x,y)+d_k(x,d_{j-k}(u,v)y).$$

Letting $v\to\overline{v}\star\overline{w}$, and then letting $u\to v\to v\to w\to u$, these give

$$[Q(u,v,w),d_k(x,y)]=d_k(Q(u,v,w)x,y)+d_k(x,Q(u,v,w)y) \qquad (5.15)$$

when we further sum over $j\equiv 0,1,2$ and note that $Q(u,v,w)$ is totaly symmetric in u,v,w. Finally, summing over k,it gives Eq.(5.14b).

Proposition 5.11

Let A^* be a pre-structurable algebra over the field F of charachterictic $\neq 2,\neq 3$. If A^* is power-associative, then A^* is structurable.

Proof

For any $a\in A^*$ satisfying $\overline{a}=a$, $(i.e, a\in S)$. we calculate

$$Q(a,a,a)a=[a,a\star a^2]^\star+3\{a^2\star a^2-a\star(a^2\star a)\} \qquad (5.16a)$$

$$=[a^2\star a,a]^\star+3\{a^2\star a^2-(a\star a^2)\star a\} \qquad (5.16b)$$

where we have set $a^2=a\star a$ and $[x,y]^\star\equiv x\star y-y\star x$. Note that Eq.(5.16a) follows immediately from Eq.(5.11) by setting $x=y=z=a$, while Eq.(5.16b) results from taking the involution of Eq.(5.16a). Then, if A^* is power-associative, Eq.(5.16) implies $Q(a,a,a)a=0$. Therefore, linearizing the relation, we obtain $Q(x,y,z)w=0$ for $x,y,z,w\in S$, since we are assuming the field F to be of charachteristic $\neq 2,\neq 3$. Together with Theorem 5.6, this shows A^* to be structurable. □

Proposition 5.12([O.05])

If a pre-sturucturable algebra A^* possesses a symmetric bi-linear non-degenerate form $<\cdot|\cdot>$ satisfying

$$<\overline{x}|y\star z>=<\overline{y}|z\star x>=<\overline{z}|x\star y>,$$

then A^* is structurable.(see Remark 4.16).

Proposition 5.13

Let A^* be a pre-structurable algebra and set $A_0 = \{x | x \in A^*, \text{and } Q(u,v,w)x = 0 \text{ for any } u,v,w \in A^*\}$, then A_0 is a structurable algebra. Moreover A_0 contains a structurable sub-algebra generated by the unit element e and members of H assuming $2 \neq 0$.

Proof

First, we show that A_0 is a sub-algebra of A^* since we calculate

$$Q(u,v,w)(xy) = (Q(u,v,w)x)y + x(Q(u,v,w)y) = 0$$

for any $x, y \in A_0$ to get $xy \in A_0$, by the derivation property of $Q(v,u,w)$. Moreover, $e \in A_0$ also by Theorem 1.4. Further, if $x \in A_0$,then $\overline{x} \in A_0$ also since

$$0 = \overline{Q(u,v,w)x} = \overline{Q(u,v,w)}\overline{x} = Q(u,v,w)\overline{x}.$$

Then, these imply $d_j(x,y) \in End\, A_0$, for $x, y \in A_0$, so that A_0 is pre-structurable. Since $Q(u,v,w) = 0$ restricted to A_0, this proves A_0 to be structurable. The fact that A_0 contains a structurable sub-algebra generated by e and H follows from Theorem 5.6. □

We can prove the converse statement of Theorem 5.6

Theorem 5.14

Let A^* be a unital involutive algebra satisfying

(i) $Q(x,y,z)w$ is totally symmetric in $x, y, z, w \in A^*$.

(ii) $Q(x,y,z) = 0$ identically whenever at least one of x, y and z is a element of H.

(iii) The validity of Eq.(sk).

Then A^* is pre-structurable.

In order to prove this Theorem, we note the following

Lemma 5.15

Under the conditions for A^* given in Theorem 5.14, we have

$$\overline{Q(x,y,z)} = B(x,y,z) - C(y,x,z) - C(z,x,y) - C^{'}(z,y,x). \qquad (5.17)$$

Proof

Eq.(5.11) leads to

$$\begin{aligned} Q(x,y,z)w \ &= w \star \{\overline{x} \star (\overline{z} \star \overline{y})\} - \{w \star (\overline{x} \star \overline{z})\} \star \overline{y} \\ &-w \star \{(y \star z) \star x\} + (w \star y) \star (z \star x) \\ &-\overline{z} \star \{(\overline{y} \star \overline{x}) \star w\} + (\overline{z} \star \overline{y}) \star (\overline{x} \star w) \\ &-x \star \{(y \star z) \star w\} + (x \star y) \star (z \star w). \end{aligned}$$

Taking the involution of this relation, we have

$$\overline{Q(x,y,z)w} = \{B(z,\overline{x},\overline{w}) - C(\overline{x},z,\overline{w}) - C(\overline{w},z,\overline{x}) - C^{'}(\overline{w},\overline{x},z)\}y. \qquad (5.18)$$

The left-hand side is rewritten as

$$\overline{Q(x,y,z)w} = \overline{Q(x,\overline{y},z)w} = \overline{Q(x,w,z)\overline{y}} = \overline{Q(x,w,z)}y$$

since $Q(x, y - \overline{y}, z) = 0$. Therefore, Eq.(5.18) is rewritten as

$$\overline{Q(x,w,z)} = B(z,\overline{x},\overline{w}) - C(\overline{x},z,\overline{w}) - C(\overline{w},z,\overline{x}) - C^{'}(\overline{w},\overline{x},z).$$

Letting $x \to \overline{x}$ and $w \to \overline{w}$, and noting $Q(\overline{x},\overline{w},z) = Q(x,w,z)$ this yields

$$\overline{Q(x,w,z)} = B(z,x,w) - C(x,z,w) - C(w,z,x) - C^{'}(w,x,z).$$

Changing $w \to z \to x \to y$, this gives Eq.(5.17).□

Since $Q(x,y,z)$ is totally symmetric in x, y and z, Eq.(5.17) immediately gives

(1)

$$B(x,y,z) - B(x,z,y) = C^{'}(z,y,x) - C^{'}(y,z,x), \qquad (5.19)$$

and

(2)

$$B(x,y,z) - B(y,x,z) =$$

$$C(y,x,z) + C(z,x,y) - C(x,y,z) - C(z,y,x) + C^{'}(z,y,x) - C^{'}(z,x,y) \quad (5.20)$$

from $Q(x,y,z) = Q(x,z,y)$ for Eq.(5.19) and $Q(x,y,z) = Q(y,x,z)$ for Eq.(5.20). Moreover, letting $x \leftrightarrow z$ in Eq.(5.19) and adding it to Eq.(5.20), we obtain

$$\{B(x,y,z) - B(y,x,z) + B(z,y,x) - B(z,x,y)\}w$$

$$\begin{aligned} &= -\{C(x,y,z) - C(y,x,z) + C(z,y,x) - C(z,x,y)\}w \\ &\quad +\{C'(x,y,z) - C'(y,x,z) + C'(z,y,x) - C'(z,x,y)\}w \\ &= -\{C(x,y,z) - C(y,x,z) + C(x,y,z) - C(z,x,y)\}(w - \overline{w}) \end{aligned} \tag{5.21}$$

since $C'(x,y,z)w = C(x,y,z)\overline{w}$.

Further, if Eq.(sk) holds, we have (see,[A-F,93])

$$C(x,y,z)(w - \overline{w}) = B(x,y,z)(w - \overline{w}) \tag{5.22}$$

when we calculate (with $s = w - \overline{w}$),

$$\begin{aligned} C(x,y,z)s &= \{x \star (\overline{y} \star \overline{s})\} \star z - (x \star \overline{y}) \star (\overline{s} \star z) \\ &= -\{x \star (\overline{y} \star s)\} \star z + (x \star \overline{y}) \star (s \star z) \\ &= \{[x, \overline{y}, s] - (x \star \overline{y}) \star s\} \star z + (x \star \overline{y}) \star (s \star z) \\ &= [x, \overline{y}, s] \star z - [x \star \overline{y}, s, z] = [s, x, \overline{y}] \star z + [s, x \star \overline{y}, z] \\ &= \{(s \star x) \star \overline{y} - s \star (x \star \overline{y})\} \star z + \{s \star (x \star \overline{y})\} \star z - s \star \{(x \star \overline{y}) \star z\} \\ &= \{(s \star x) \star \overline{y}\} \star z - s \star \{(x \star \overline{y}) \star z\} = B(x,y,z)s. \end{aligned}$$

Then, Eq.(5.21) is rewritten as

$$\{B(x,y,z) - B(y,x,z) + B(z,y,x) - B(z,x,y)\}w$$

$$= -\{B(x,y,z) - B(y,x,z) + B(z,y,x) - B(z,x,y)\}(w - \overline{w}). \tag{5.23}$$

However, Eq.(5.19) for $(x \leftrightarrow z)$, gives also

$$\begin{aligned} \{B(z,y,x) - B(z,x,y)\}(w - \overline{w}) &= \{C'(x,y,z) - C'(y,x,z)\}(w - \overline{w}) \\ &= -\{B(x,y,z) - B(y,x,z)\}(w - \overline{w}) \end{aligned}$$

or

$$\{B(x,y,z) - B(y,x,z) + B(z,y,x) - B(z,x,y)\}(w - \overline{w}) = 0$$

which yields Eq.(B),i.e.

$$B(x,y,z) - B(y,x,z) + B(z,y,x) - B(z,x,y) = 0 \qquad (B)$$

in view of Eq.(5.23).

We next set $x = e$ in Eq.(5.19) to obtaion

$$\{B(e,y,z) - B(e,z,y)\}w = \{C'(z,y,e) - C'(y,z,e)\}w$$

or equivalently

$$[w,\overline{y},z] - [w,\overline{z},y] = [y,\overline{z},w] - [z,\overline{y},w]$$

which is one of Eq.(A.1), if we change variables suitably. Then together with Eq.(5.9), we find the validity of Eq.(5.6a)i.e.

$$A(x,y,z) - A(y,z,x) = A(z,x,y) - A(z,y,x). \qquad (A)$$

Since Eq.(5.6b) is nothing but Eq.(5.19), then Lemma 5.3 show that the A^* is an almost pre-structurable algebra. But A^* is unital by assumption and these prove A^* to be pre-structurable, This completes the proof of Theorem 5.14.□

The special case of $Q(x,y,z) = 0$ identically in Lemma 5.15 immediately reproduces (iii) of Theorem 5.5 of [A-F,93] by giving

$$B(x,y,z) = C(y,x,z) + C(z,x,y) + C'(z,y,x). \qquad (X)$$

Theorem 5.16

A necessary and sufficient condition that a unitary involutive algebra A^* being structurable is the validily of Eq.(sk) and Eq.(X) (or equivalently $Q(x,y,z) = 0$).

Remark 5.17

Many interesting unital involution algebra containing Jordan and alternative algebras are structurable. It is rather hard to find examples of a simple pre-structurable but not structurable algebra.

6. Kantor Triple System and A-ternary Algebra

Let V be a vector space over a field F, equipped with a tri-linear map

$$V \otimes V \otimes V \to V$$

$$x \otimes y \otimes z \to xyz. \tag{6.1}$$

If the triple product xyz satisfies

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz) \tag{6.2}$$

for any $u, v, x, y, z \in V$, then (V, xyz) is called a generalized Jordan triple system. Moreover, if it satisfies

$$xyz = zyx, \tag{6.3}$$

then (V, xyz) defines a Jordan triple system [J.68]. It is often more convenient to introduce a multiplication operator $L(x, y) \in \text{End } V$ by

$$L(x, y)z := xyz. \tag{6.4}$$

Then, Eq.(6.2) is equivalent to a Lie algebra relation of

$$[L(u, v), L(x, y)] = L(uvx, y) - L(x, vuy). \tag{6.5}$$

Moreover, suppose that $K(x, y) \in \text{End } V$ given by

$$K(x, y)z = xzy - yzx \tag{6.6}$$

satisfies

$$K(K(u, v)x, y) = L(y, x)K(u, v) + K(u, v)L(x, y). \tag{6.7}$$

Then (V, xyz) is called a Kantor triple system ([Kan.73]). Note that the Jordan triple system is a Kantor triple system with $K(x, y) = 0$.

Also it is known (see Eq.(7.6)) that the condition Eq.(6.7) is equivalent to

$$K(xyz, w) - K(xyw, z) = -K(x, K(z, w)y), \tag{6.8}$$

if Eq.(6.5) holds valid.

A main purpose of this section is to note that a structurable algebra is intimately related to the Kantor triple system as is indicated in the following Theorem (see [F.94],[K-O.10]):

Theorem 6.1

Let A^* be a structurable algeba over a field F of charachteristic $\neq 2$. If we define a triple product xyz in the vector space of A^* by

$$xyz := (z \star \overline{y}) \star x - (z \star \overline{x}) \star y + (x \star \overline{y}) \star z, \tag{6.9}$$

then (A^*, xyz) is a Kantor triple system such that it satisfies

$$eex = x, \text{ and } exe + 2xee = 3x \tag{6.10}$$

for the unit element e of A^*. Conversely if (A^*, xyz) is a Kantor triple system over a field F of charachteristic $\neq 2, \neq 3$, satisfying Eq.(6.10) for a privileged element e of A^*, and if we introduce a mapping $x \to \overline{x}$ and a bi-linear product $x \star y$ in A^* by

$$\bar{x} := 2x - xee, \tag{6.11a}$$

$$x \star y := \bar{x}ey - \bar{x}\bar{y}e + yex, \tag{6.11b}$$

then $(A^*, x \star y)$ is a structurable algebra with the unit element e and the involution map $x \to \overline{x}$.

First, we shall prove here a slightly weaker theorem in the following.

Theorem 6.2([O.05])

Let A^* be normal Lie-related triality algebra (see Def.1.6). Then, the triple product in A^* defined by

$$xyz := k\{l(x \star \bar{y} + y \star \bar{x}) - d_0(x, y) - d_2(\bar{x}, \bar{y})\}z \tag{6.12}$$

for $k \in F$, $(k \neq 0)$, leads to a generalized Jordan triple system (A^*, xyz).

For a proof of this Theorem, we need the following Lemma.

Lemma 6.3

Let A^* be a pre-normal Lie related triality algebra. If we set

$$D_0(x, y) := d_0(x, y) + d_2(\bar{x}, \bar{y}), \tag{6.13a}$$

then it satisfies

$$[D_0(u, v), D_0(x, y)] = D_0(D_0(u, v)x, y) + D_0(x, D_0(u, v)y). \tag{6.13b}$$

We also have

$$[d_{3-j}(\bar{x},\bar{y}) + d_{j+2}(x,y), l(z)] = l((d_{j+1}(x,y) + d_{2-j}(\bar{x},\bar{y}))z), \tag{6.14a}$$

and

$$\{d_{3-j}(\bar{x},\bar{y}) - d_{j+2}(x,y), l(z)\}_{(+)} = l((d_{j+1}(x,y) - d_{2-j}(\bar{x},\bar{y}))z) \tag{6.14b}$$

where we have set

$$[X,Y] = XY - YX,$$
$$\{X,Y\}_+ = XY + YX$$

for $X, Y, \in \text{End } A^*$.

Proof

By Eq.(1.20), we calculate

$$[D_0(u,v), D_0(x,y)]$$
$$= [d_0(u,v) + d_2(\overline{u},\overline{v}), d_0(x,y) + d_2(\overline{x},\overline{y})]$$
$$= d_0((d_0(u,v) + d_2(\overline{u},\overline{v}))x, y) + d_0(x, (d_0(u,v) + d_2(\overline{u},\overline{v}))y)$$
$$+ d_2((d_1(u,v) + d_0(\overline{u},\overline{v}))\overline{x},\overline{y}) + d_2(x, (d_1(u,v) + d_0(\overline{u},\overline{v}))\overline{y}).$$

Also, we note

$$D_0(D_0(u,v)x,y) = d_0(D_0(u,v)x,y) + d_2(\overline{D_0(u,v)x},\overline{y})$$

$$= d_0((d_0(u,v) + d_2(\overline{u},\overline{v}))x, y) + d_2((d_0(\overline{u},\overline{v}) + d_1(u,v))\overline{x},\overline{y})$$

and

$$D_0(x, D_0(u,v)y) = d_0(x, D_0(u,v)y) + d_2(\overline{x}, \overline{D_0(u,v)y})$$
$$= d_0(x, (d_0(u,v) + d_2(\overline{u},\overline{v}))y) + d_2(\overline{x}, (d_0(\overline{u},\overline{v}) + d_1(u,v))\overline{y})$$

where we noted Eq.(1.23). These prove Eqs.(6.13b).

We next rewrite the triality relation Eq.(1.41d) as

$$d_{3-j}(\overline{x},\overline{y})(z \star w) = (d_{j+1}(x,y)z) \star w + z \star (d_{j+2}(u,v)w)$$

which yield

$$d_{3-j}(\overline{x},\overline{y})l(z) = l(d_{j-1}(x,y)z) + l(z)d_{j+2}(\overline{x},\overline{y}). \tag{6.15}$$

Letting $x \to \bar{x}$ and $y \to \bar{y}$ with $j \to 1-j$, this is rewritten as

$$d_{j+2}(x,y)l(z) = l(d_{2-j}(\overline{x},\overline{y})z) + l(z)d_{3-j}(x,y).$$

Adding or sub-tracting both relations, we obtain Eqs.(6.14).□

After these preparations, we will now proceed to the proof of Theorem 6.2. First, choosing $j=0$ in Eq.(6.14a) and letting $x \to \overline{x}$ and $y \to \overline{y}$, it yields

$$[D_0(x,y), l(z)] = l((d_1(\overline{x},\overline{y}) + d_2(x,y))z). \tag{6.16}$$

For simplicity, we set

$$s = u \star \overline{v} + v \star \overline{u}, \; t = x \star \overline{y} + y \star \overline{x} \tag{6.17}$$

so that we can write

$$\begin{aligned} L(u,v) &= k\{l(s) - D_0(u,v)\} \\ L(x,y) &= k\{l(t) - D_0(x,y)\}, \end{aligned} \tag{6.18}$$

and calculate

$$\begin{aligned} &[L(u,v), L(x,y)] = k^2[l(s) - D_0(u,v), l(t) - D_0(x,y)] \\ &= k^2\{[l(s),l(t)] - [D_0(u,v),l(t)] + [D_0(x,y),l(s)] + [D_0(u,v),D_0(x,y)]\} \\ &= k^2\{[l(s),l(t)] - l((d_1(\overline{u},\overline{v}) + d_2(u,v))t) + l((d_1(\overline{x},\overline{y}) + d_2(x,y))s) \\ &\quad + D_0(D_0(u,v)x,y) + D_0(x,D_0(u,v)y)\}. \end{aligned} \tag{6.19}$$

Also

$$\begin{aligned} &L(L(u,v)x,y) \\ &= k^2\{l(\,(l(s) - D_0(u,v))x \star \overline{y} + y \star \overline{((l(s) - D_0(u,v))x)}\,) \\ &\quad - D_0((l(s) - D_0(u,v))x,y)\} \\ &= k^2\{l((s \star x) \star \overline{y} + y \star (\overline{x} \star s)) - l(D_0(u,v)x \star \overline{y} + y \star \overline{D_0(u,v)x}) \\ &\quad - D_0(s \star x, y) + D_0(D_0(u,v)x,y)\} \end{aligned}$$

and

$$\begin{aligned} &L(x, L(v,u)y) \\ &= k(l(x \star \overline{L(v,u)y} + L(v,u)y \star \overline{x}\} - D_0(x, L(v,u)y)\} = \end{aligned}$$

$$k^2\{l\{x\star\overline{((l(s)+D_0(u,v))y)}+(l(s)+D_0(u,v))y\star\overline{x}\}-D_0(x,(l(s)+D_0(u,v))y)\}$$
$$=k^2\{l(x\star(\overline{y}\star s)+(s\star y)\star\overline{x})+l(x\star\overline{D_0(u,v)y})+l(D_0(u,v)y\star\overline{x})$$
$$-D_0(x,s\star y)-D_0(x,D_0(u,v)y)\}$$

where we noted $\overline{s}=s$ an $L(v,u)=l(s)-D_0(v,u)=l(s)+D_0(u,v)$. Then setting

$$R=[L(u,v),L(x,y)]-L(L(u,v)x,y)+L(x,L(v,u)y),$$

we can rewrite it as

$$R=k^2\{[l(s),l(t)]+l(w)+D_0(s\star x,y)-D_0(x,s\star y)\}, \tag{6.20}$$

with

$$\begin{aligned} w=&-(d_1(\overline{u},\overline{v})+d_2(u,v))t+(d_1(\overline{x},\overline{y})+d_2(x,y))s\\ &-(s\star x)\star\overline{y}-y\star(\overline{x}\star s)+x\star(\overline{y}\star s)+(s\star y)\star\overline{x}+J, \end{aligned} \tag{6.21}$$

where we have set further

$$J=(D_0(u,v)x)\star\overline{y}+y\star\overline{D_0(u,v)x}+x\star\overline{D_0(u,v)y}+(D_0(u,v)y)\star\overline{x}.$$

We calculate J to be

$$\begin{aligned} J\ &=(d_0(u,v)+d_2(\overline{u},\overline{v}))x\star y+y\star(d_0(\overline{u},\overline{v})+d_1(u,v))\overline{x}\\ &+x\star(d_0(\overline{u},\overline{v})+d_1(u,v))\overline{y}+\underline{(d_0(u,v)+d_2(\overline{u},\overline{v}))y\star\overline{x}}\\ &=\overline{d_2(u,v)}(x\star\overline{y})+\overline{d_1(\overline{u},\overline{v})}(x\star y)+\overline{d_2(u,v)}(y\star x)+\overline{d_1(\overline{u},\overline{v})}(y\star\overline{x})\\ &=(d_1(\overline{u},\overline{v})+d_2(u,v))(x\star\overline{y}+y\star\overline{x})=(d_1(\overline{u},\overline{v})+d_2(u,v))t. \end{aligned}$$

by the triality relation Eq.(1.39c) and Eq.(1.39g). This then leads to $w=0$ since we calculate

$$\begin{aligned} w\ &=-(d_1(\overline{u},\overline{v})+d_2(u,v))t+(d_1(\overline{x},\overline{y})+d_2(x,y))s\\ &+\{-r(\overline{y})r(x)-l(y)l(\overline{x})+l(x)l(\overline{y})+r(\overline{x})r(y)\}s\\ &+(d_1(\overline{u},\overline{v})+d_2(u,v))t=0. \end{aligned}$$

Moreover, we note

$$D_0(s\star x,y)-D_0(x,s\star y)=d_0(s\star x,y)+d_2(\overline{s\star x},\overline{y})-d_0(x,s\star y)-d_2(\overline{x},\overline{s\star y})=$$

$$-d_0(y,\overline{\overline{x}\star s})-d_2(\overline{x},\overline{\overline{y}\star s})-d_0(x,\overline{\overline{y}\star s})-d_2(\overline{y},\overline{s\star x})=$$

$$-\{Q(y,\bar{x},s) - d_1(s,\overline{y\star\bar{x}})\} - \{Q(x,\bar{y},s) - d_1(s,\overline{x\star\bar{y}})\} =$$

$$d_1(s,x\star\bar{y}) + d_1(s,y\star\bar{x}) = d_1(s,x\star\bar{y}+y\star\bar{x}) = d_1(s,t) = \;= -[l(s),l(t)]$$

where we used $Q(x,y,z) = 0$ together with $\bar{s} = s$ and $\bar{t} = t$. Therefore we have $R = 0$ by Eq.(6.20), completing the proof of Theorem 6.2.□

Remark 6.4

If we rewrite Eq.(6.12) as

$$xyz = k\{(x\star\bar{y} + y\star\bar{x})\star z + (z\star\bar{y})\star x - (z\star\bar{x})\star y - d_0(x,y)z\},$$

and note Eq.(1.39d)i.e.

$$d_0(x,y)z + d_0(y,z)x + d_0(z,x)y = 0,$$

this gives

$$K(x,y) = k\{r(y)r(\bar{x}) - r(x)r(\bar{y}) + l(x\star\bar{y} - y\star\bar{x}) - d_0(x,y)\}. \qquad (6.22)$$

Hereafter in this section, we assume A^* to be structurable over the field F of charachterictic $\neq 2$. Then choosing $k = \frac{1}{2}$ and noting

$$d_0(x,y) = r(y)r(\bar{x}) - r(x)r(\bar{y}) - l(x\star\bar{y} - y\star\bar{x}),$$

Eq.(6.22) gives

$$K(x,y) = l(x\star\bar{y} - y\star\bar{x}) \qquad (6.23)$$

while Eq.(6.12) reproduces Eq.(6.9). We can then prove the validity of Eq.(6.7) as follows: Setting now

$$s = u\star\bar{v} - v\star\bar{u} = w - \bar{w},\ (w = u\star\bar{v}), \qquad (6.24)$$

we have $K(u,v) = l(s)$ so that Eq.(6.7) becomes

$$K(s\star x,y)z = yx(s\star z) + s\star(xyz)$$

or

$$\{(s\star x)\star\bar{y} - y\star\overline{(s\star x)}\}\star z =$$

$$\{(s\star z)\star\bar{x}\}\star y - \{(s\star z)\star\bar{y}\}\star x + (y\star\bar{x})\star(s\star z) + s\star\{(z\star\bar{y})\star x - (z\star\bar{x})\star y + (x\star\bar{y})\star z\}$$

by Eq.(6.9) and (6.23). This relation is rewriten as

$$\{B(x,y,z) - B(z,x,y) + B(z,y,x) - C(y,x,z)\}s = 0$$

which is satisfied in view of Eqs.(B) and (5.22). This proves that (A^*, xyz) is a Kantor triple system.

Moreover,Eq.(6.10) holds valid by Eq.(6.9). However, the converse statement that any Kantor triple system (A^*, xyz) satisfying Eq.(6.10) will give a structurable algebra requires more calculations. See [F.94] or [K-O.10] for its proof.

Remark 6.5

The structurable algebra has been originally defined by Allison ([A.78]) in terms of the Kantor triple system as in Theorem 6.1. We redefined it in this note as in [K-O.14]. It is often more appropriate to consider a triple $(A^*, x \star y, xyz)$ by defining:

Def.6.6

A triple $(A^*, x \star y, xyz)$ with a bi-linear product $x \star y$ and a triple product xyz in a vector space A^* is called an Allison ternary algebra or simply A-ternary algebra, provided that we have

(1) $(A^*, x \star y)$ is a structurable algebra

(2) (A^*, xyz) is a Kantor triple system

(3) The triple product xyz is expressed in terms of the bi-linear product by

$$xyz = (x \star \bar{y}) \star z + (z \star \bar{y}) \star x - (z \star \bar{x}) \star y.$$

Note that both the bi-linear product $x \star y$ and the tri-linear one xyz are intimately related to each other as in Theorem 6.1. Other example of such triple are (α, β, γ) ternary algebra ([K-O.10]) based upon unital (α, β, γ) ternary system as well as some balanced $(-1, -1)$FKTS (see [E-K-O.03 and 05]).

For the A-ternary algebra, we note that we have

$$\begin{aligned} K(x,y) &= d_2(\bar{x}, \bar{y}) - d_0(x,y) = l(x \star \bar{y} - y \star \bar{x}) \\ L(x,y) + L(y,x) &= l(x \star \bar{y} + y \star \bar{x}) \\ L(x,y) - L(y,x) &= -d_0(x,y) - d_2(\bar{x}, \bar{y}) \end{aligned} \tag{6.25}$$

which determine $K(x,y)$ and $L(x,y)$ in terms of $d_j(x,y)$ and $l(x)$.

Conversly, we can express

$$\begin{aligned}
&l(x+\bar{x}) = L(e,x) + L(x,e)\\
&l(x-\bar{x}) = K(x,e)\\
&d_0(x,y) = \tfrac{1}{2}\{L(y,x) - L(x,y) - K(x,y)\}\\
&d_2(x,y) = \tfrac{1}{2}\{K(\bar{x},\bar{y}) - L(\bar{x},\bar{y}) + L(\bar{y},\bar{x})\}\\
&d_1(x,y) - d_1(\bar{x},\bar{y}) = -K(\bar{x},\bar{y}) + K(x,y) \qquad (6.26)\\
&d_1(x,y) + d_1(\bar{x},\bar{y}) = L(y,x) - L(x,y) + L(e,\bar{x}\star y) - L(\bar{x}\star y,e).
\end{aligned}$$

These relations will be used to prove some results in the next section.

7. Lie algebras and superalgebras associated with (ε,δ) Freudenthal -Kantor Triple System (FKTS)

In this section, we first note that the Kantor triple system is a special case of a more general (ε,δ) Freudenthal-Kantor triple system [Y-O.84], (see also [Kam.87] for many earlier references on the subject) and second that we can construct Lie or Lie superalgebra out of these triple systems.

Let V be a vector space over a field F with a triple product xyz. Let the multiplication operators $L(x,y)$, and $K(x,y) \in \text{End}\, V$ be given by

$$L(x,y)z := xyz,\ K(x,y)z := xzy - \delta yzx. \qquad (7.1)$$

If they satisfy

$$[L(u,v), L(x,y)] = L(uvx,y) + \varepsilon L(x,vuy) \qquad (7.2)$$

and

$$K(K(u,v)x,y) = L(y,x)K(u,v) - \varepsilon K(u,v)L(x,y) \qquad (7.3)$$

for any $u,v,x,y,\in V$, then, we call the triple system (V,xyz) be a (ε,δ) Freudenthal-Kantor triple system [Y-O.84], where ε and δ are constants with values either 1 or -1. Then, comparing these with Eq.(6.5),(6.6), and (6.7), we see that the Kantor triple system is precisely $(-1,1)$ FKTS. Before going into further details, we note that Eq.(7.1) implies

$$K(y,x) = -\delta K(x,y) \qquad (7.4)$$

while Eq.(7.2) is equivalent to the validity of

$$uv(xyz) = (uvx)yz + \varepsilon x(vuy)z + xy(uvz). \qquad (7.5)$$

First, we note that Eq.(7.3) can be replaced by

$$K(K(u,v)x,y) = K(yxu,v) + K(u,yxv) \tag{7.6}$$

under the validity of Eq.(7.2) when we note

Lemma 7.1

If Eq.(7.2) holds, we then have

$$L(y,x)K(u,v) - \varepsilon K(u,v)L(x,y) = K(yxu,v) + K(u,yxv). \tag{7.7}$$

Proof

We calculate

$$\begin{aligned}&L(y,x)K(u,v)z - \varepsilon K(u,v)L(x,y)z\\&= yx(uzv - \delta vzu) - \varepsilon\{u(xyz)v - \delta v(xyz)u\}.\end{aligned}$$

Moreover, we note

$$yx(uzv) = (yxu)zv + \varepsilon u(xyz)v + uz(yxv)$$

by Eq.(7.5) so that

$$\begin{aligned}&L(y,x)K(u,v)z - \varepsilon K(u,v)L(x,y)z\\&= (yxu)zv + \varepsilon u(xyz)v + uz(yxv)\\&-\delta(yxv)zu - \varepsilon\delta v(xyz)u - \delta vz(yxu)\\&-\varepsilon u(xyz)v + \varepsilon\delta v(xyz)u\\&= (yxu)zv + uz(yxv) - \delta(yxv)zu - \delta vz(yxu)\\&= \{(yxu)zv - \delta vz(yxu)\} + \{uz(yxv) - \delta(yxv)zu)\}\\&= K(yxu,v)z + K(u,yxv)z,\end{aligned}$$

which yields Eq.(7.7). □

Proposition 7.2

Let (V, xyz) be a (ε,δ) FKTS. We then have

$$\begin{aligned}&K(u,v)K(x,y)\\&= \varepsilon\delta L(K(u,v)y,x) - \varepsilon L(K(u,v)x,y)\end{aligned} \tag{7.8a}$$

$$= L(v, K(x,y)u) - \delta L(u, K(x,y)v) \qquad (7.8b)$$

for any $u, v, x, y \in V$.

Proof

First, let us define

$$K(V,V) := \text{span}\{K(x,y), \forall, x, y \in V\}. \qquad (7.9)$$

Then, for any $\sigma \in K(V,V)$, Eq.(7.3) gives

$$K(\sigma x, y) = L(y,x)\sigma - \varepsilon\sigma L(x,y) \qquad (7.10)$$

so that

$$(\sigma x)zy - \delta yz(\sigma x) = yx(\sigma z) - \varepsilon\sigma(xyz)$$

which is rewritten as

$$\sigma(xyz) = -\varepsilon(\sigma x)zy + \varepsilon yx(\sigma z) + \varepsilon\delta yz(\sigma x) \qquad (7.11)$$

since $\varepsilon^2 = 1$. We then calculate

$$\sigma K(x,z)y = \sigma(xyz) - \delta\sigma(zyx)$$

$$= -\varepsilon(\sigma x)zy + \varepsilon yx(\sigma z) + \varepsilon\delta yz(\sigma x) + \varepsilon\delta(\sigma z)xy - \varepsilon\delta yz(\sigma x) - \varepsilon\delta^2 yx(\sigma z)$$

$$= -\varepsilon(\sigma x)zy + \varepsilon\delta(\sigma z)xy = -\varepsilon L(\sigma x, z)y + \varepsilon\delta L(\sigma z, x)y$$

which gives

$$\sigma K(x,z) = -\varepsilon L(\sigma x, z) + \varepsilon\delta L(\sigma z, x).$$

Letting $z \to y$ and choosing $\sigma = K(u,v)$, this leads to Eq.(7.8a).

In order to prove Eq.(7.8b), we note

$$[L(u,v), L(x,y)] = L(uvx, y) + \varepsilon L(x, vuy)$$
$$= -[L(x,y), L(u,v)] = -L(xyu, v) - \varepsilon L(u, yxv)$$

so that

$$L(uvx, y) + \varepsilon L(x, vuy) + L(xyu, v) + \varepsilon L(u, yxv) = 0. \qquad (7.12)$$

Letting $x \leftrightarrow u$, it also gives

$$L(xvu, y) + \varepsilon L(u, vxy) + L(uyx, v) + \varepsilon L(x, yuv) = 0.$$

Multiplying δ and subtracting it from Eq.(7.12), we find

$$L(uvx - \delta xvu, y) + \varepsilon L(x, vuy - \delta yuv) +$$
$$\varepsilon L(u, yxv - \delta vxy) + L(xyu - \delta uyx, v) = 0,$$

or

$$L(K(u,x)v, y) + \varepsilon L(x, K(v,y)u) + \varepsilon L(u, K(y,v)x) + L(K(x,u)y, v) = 0.$$

Changing $x \leftrightarrow v$ and noting $K(y,x) = -\delta K(x,y)$, this is rewritten as

$$L(K(u,v)x, y) - \delta L(K(u,v)y, x) = \varepsilon\delta L(u, K(x,y)v) - \varepsilon L(v, K(x,y)u)$$

which proves Eq.(7.8b)□

We can then prove the following (see [K-M-O.10]).

Corollary 7.3

$K(x,y)$'s satisfy

(1)

$$\begin{aligned} &K(z,w)K(x,y)K(u,v) + K(u,v)K(x,y)K(z,w) \\ &= K(K(z,w)K(x,y)u, v) + K(u, K(z,w)K(x,y)v) \qquad (7.13a) \\ &= \varepsilon\delta K(K(z,w)x, K(u,v)y) + \varepsilon\delta K(K(u,v)x, K(z,w)y) \end{aligned}$$

(2)

$$\begin{aligned} &[[K(z,w), K(u,v)], K(x,y)] = \\ &-\varepsilon K(x, [K(z,w), K(u,v)]y) - \varepsilon K([K(z,w), K(u,v)]x, y) \qquad (7.13b) \end{aligned}$$

for any $u, v, x, y, z, w \in V$. Especially, if we introduce two triple products in $K(V,V)$ by

(a)

$$\{K_1, K_2, K_3\} := K_1K_2K_3 + K_3K_2K_1 \qquad (7.14a)$$

(b)

$$[K_1, K_2, K_3] := [[K_1, K_2], K_3] \qquad (7.14b)$$

then they define a Jordan triple system for $\{K_1, K_2, K_3\}$ and a Lie triple system for $[K_1, K_2, K_3]$, respectivaly for $K_1, K_2, K_3 \in K(V, V)$.

Remark 7.4

Let $V = A^*$ be a structurable algebra over the field F of charachteristic $\neq 2$. Then Eq.(6.23) implies $K(A^*, A^*) = l(H)$ since $x \star \overline{e} - e \star \overline{x} = (x - \overline{x}) \in H$ where $H = \{\overline{x} = -x,\ x \in A^*\}$. Then Corollary 7.3 implies that $l(H)$ admitts both Jordan triple system and Lie triple system for $K_j \in l(H)$ by Eqs.(7.14) with $(\varepsilon, \delta) = (-1, 1)$.

Eqs.(7.8) enables us to prove the following theorem:

Theorem 7.5

Let $T(\varepsilon, \delta) := (V, xyz)$ be a (ε, δ)FKTS. Suppose that $J \in \text{End } V$ satisfies

(1)

$$J(xyz) = (Jx)(Jy)(Jz) \tag{7.15b}$$

(2)

$$J^2 = -\varepsilon\delta \text{id} \tag{7.15b}$$

then a new triple product defined by

$$[x, y, z] := x(Jy)z - \delta y(Jx)z + \delta x(Jz)y - y(Jz)x \tag{7.10}$$

satisfies

(1)

$$[x, y, z] := -\delta[y, x, z] \tag{7.17a}$$

(2)

$$[x, y, z] + [y, z, x] + [z, x, y] = 0 \tag{7.17b}$$

(3)

$$[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]. \tag{7.17c}$$

In other words, $[x, y, z]$ is a Lie triple system for $\delta = 1$ and an anti-Lie triple system for $\delta = -1$.

For a proof of this Theorem, the readers are reffered to [K-O.00] or [K-M-O.10]. We note that for the case of the Kantor triple system ($\varepsilon = -1, \delta = +1$), we can chose $J = id$. However, a more interesting case is to consider a larger vector space:

$$W = V \oplus V, \text{or } W = \begin{pmatrix} V \\ V \end{pmatrix}, \tag{7.18}$$

and intorduce a triple product in W by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1x_2x_3 \\ y_1y_2y_3 \end{pmatrix}. \tag{7.19}$$

Then $W = \begin{pmatrix} V \\ V \end{pmatrix} = V \oplus V$ is also a (ε, δ) FKTS, if V is a (ε, δ)FKTS. Moreover $J \in \text{End } W$ given by

$$J = \begin{pmatrix} 0, & 1 \\ -\varepsilon\delta, & 0 \end{pmatrix}, \tag{7.20}$$

satisfies the conditions of Eqs.(7.15) in W.
Then, the resulting Lie or anti-Lie triple system in W is rewritten as

$$[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}]$$

$$= \begin{pmatrix} L(u,y) - \delta L(x,v), & \delta K(u,x) \\ -\varepsilon K(v,y), & \varepsilon L(y,u) - \varepsilon\delta L(v,x) \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}. \tag{7.21}$$

Further, we define a multiplication operator $L(X_1, X_2) \in \text{End } W$ by

$$L(X_1, X_2)X_3 := [X_1, X_2, X_3] \tag{7.22}$$

and set

$$L(W, W) = \text{span}\{L(X_1, X_2), \forall X_1, X_2 \in W\}. \tag{7.23}$$

Then, for any x, y, z, u, v and $w \in V$, D given by

$$D = \begin{pmatrix} L(x,y), & \delta K(z,w) \\ -\varepsilon K(u,v), & \varepsilon L(y,x) \end{pmatrix} \in L(W, W) \tag{7.24}$$

is a derivation of the triple product $[X_1, X_2, X_3]$, by the analogue of Eq.(7.17c) when we replace $x \to X,\ y \to Y,\ z \to Z$ then, i.e. we have

$$D[X_1, X_2, X_3] = [DX_1, X_2, X_3] + [X_1, DX_2, X_3] + [X_1, X_2, DX_3]. \tag{7.25}$$

Further, we set

$$L_{\bar{1}} = W = \{X = \begin{pmatrix} x \\ y \end{pmatrix} | x, y \in V\}$$

$$L_{\bar{0}} = \{D | D \text{ is a derivation of } [X_1, X_2, X_3]\}. \tag{7.26}$$

Then

$$L = L_{\bar{0}} \oplus L_{\bar{1}} \tag{7.27}$$

is a Lie algebra for $\delta = 1$, but a Lie superalgebra for $\delta = -1$ with $L_{\bar{0}}$ and $L_{\bar{1}}$ being its even and odd part, respectively. Here, we define the commutator by (see [Y-O.84])

$$[D_1 \oplus X_1, D_2 \oplus X_2] = ([D_1, D_2] + [X_1, X_2]) \oplus (D_1 X_2 - D_2 X_1) \tag{7.28}$$

for $D_1, D_2 \in L_{\bar{0}}$ and $X_1, X_2 \in W = L_{\bar{1}}$ by

$$[D_1, D_2] := D_1 D_2 - D_2 D_1 \in L_{\bar{0}} \tag{7.29}$$

and

$$[X_1, X_2] = [\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}] = L(X_1, X_2) =$$

$$\begin{pmatrix} L(x_1, y_2) - \delta L(x_2, y_1), & \delta K(x_1, x_2) \\ -\varepsilon K(y_1, y_2), & \varepsilon L(y_2, x_1) - \varepsilon \delta L(y_1, x_2) \end{pmatrix} \in L_{\bar{0}}. \tag{7.29c}$$

However, we will restrict ourselves for a choice of the derivation to be those given by Eq.(7.24) and write Eq.(7.27) as

$$L = L(W, W) \oplus W. \tag{7.30}$$

Then, L is 5-graded Lie algebra or Lie superalgebra:

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \tag{7.31}$$

where we have set

$$L_{-2} = \text{span}\{\begin{pmatrix} 0, & 0 \\ K(x, y), & 0 \end{pmatrix} | x, y \in V\} \tag{7.32a}$$

$$L_{-1} = \text{span}\{\begin{pmatrix} 0 \\ x \end{pmatrix} | x \in V\} \tag{7.32b}$$

$$L_0 = \text{span}\{\begin{pmatrix} L(x,y), & 0 \\ 0, & \varepsilon L(y,x) \end{pmatrix} |x, y \in V\} \tag{7.32c}$$

$$L_1 = \text{span}\{\begin{pmatrix} x \\ 0 \end{pmatrix} |x \in V\} \tag{7.32d}$$

$$L_2 = \text{span}\{\begin{pmatrix} 0, & K(x,y) \\ 0, & 0 \end{pmatrix} |x, y \in V\}. \tag{7.32e}$$

Note

$$L_{\bar{0}} = L_{-2} \oplus L_0 \oplus L_2, \tag{7.33a}$$

$$L_{\bar{1}} = L_{-1} \oplus L_1. \tag{7.33b}$$

If we introduce operators θ and $\sigma(\lambda)$ for $\lambda \in F, \lambda \neq 0$ in $\text{End}(L)$ by

$$\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon y \\ \delta x \end{pmatrix} \tag{7.34a}$$

$$\sigma(\lambda) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ y/\lambda \end{pmatrix}, \tag{7.34b}$$

it is easy to see that they are automorphysm of $[W, W, W]$,i.e. we have for example

$$\theta([X_1, X_2, X_3]) = [\theta X_1, \theta X_2, \theta X_3].$$

Extending these actions to the whole of L in a natural way, we find also that they are also automorphysm of the Lie algebra or Lie superalgebra L. Moreover, they satisfy

(1) $\theta^4 = id,\ \sigma(1) = \text{id}$

(2) $\theta^2 = -\varepsilon\delta$ id for $L_{\bar{1}}$ and $\theta^2 = \text{id}$ for $L_{\bar{0}}$

(3) $\sigma(\mu)\sigma(\nu) = \sigma(\mu\nu),\ (\mu, \nu \in F, \mu\nu \neq 0)$

(4) $\sigma(\lambda)\theta\sigma(\lambda) = \theta$, for any $\lambda \in F, \lambda \neq 0$.

Conversely, for any 5-graded Lie algebra or Lie super-algebra L satisfying these conditions, $A = L_{\bar{1}}$ with a triple product given by

$$\{X, Y, Z\} := [[X, \theta Y], Z]$$

for any $X, Y, Z \in A = L_{\bar{1}}$, essentially define a (ε, δ)FKTS (see [E-K-O.13]).

After these preparations, we shall now restrict ourselves to the case of Kantor triple system derived from a structurable algebra A^*. Then, we can construct Lie algebras in the two defferent ways: one as in the present-section, and the second one from the structurable algebra as in section 3. A question will arise for relationship between these two Lie algebras. We will show next that we can construct a Lie algebra given in section 3 from that of the present section for a A-ternary algebra $(A^*, x \star y, xyz)$.

Theorem7.6

Let $(A^*, x{\star}y, xyz)$ be a A-ternary algebra. Regarding (A^*, xyz) as a Kantor triple system, we contruct a Lie algebra

$$L = L(W, W) \oplus W \tag{7.35}$$

as in Eqs(7.28), and (7.29).

For any non-zero constant $\alpha, \beta, k \in F$, we introduce rational numbers γ_1/γ_3 and γ_2/γ_3 for $\gamma_j \in F$ by

(1)

$$\gamma_2/\gamma_3 = -2\alpha\beta \tag{7.36}$$

(2)

$$(\gamma_3)^2/\gamma_1\gamma_2 = -k^2,$$

and introduce $\rho_j(x)$ and $T_j(x, y)$ $(j = 0, 1, 2)$ by

$$\rho_1(x) = \begin{pmatrix} \alpha x \\ \beta x \end{pmatrix}, \ \rho_2(x) = \begin{pmatrix} k\alpha\bar{x} \\ -k\beta\bar{x} \end{pmatrix}, \tag{7.37a}$$

$$\rho_0(x) = (\frac{k\gamma_1}{\gamma_2}) \begin{pmatrix} \alpha\beta l(x+\bar{x}), & \alpha^2 l(x-\bar{x}) \\ -\beta^2 l(x-\bar{x}), & -\alpha\beta l(x+\bar{x}) \end{pmatrix} \tag{7.37b}$$

and

$$T_j(x, y) =$$

$$-\frac{\gamma_3}{\gamma_2} \begin{pmatrix} \alpha\beta(d_{j+1}(x,y) + d_{1-j}(\bar{x},\bar{y})), & \alpha^2(d_{j+1}(x,y) - d_{1-j}(\bar{x},\bar{y})) \\ \beta^2(d_{j+1}(x,y) - d_{1-j}(\bar{x},\bar{y})), & \alpha\beta(d_{j+1}(x,y) + d_{1-j}(\bar{x},\bar{y})) \end{pmatrix} \tag{7.37c}$$

for $j = 0, 1, 2$.

We then can show that both $\rho_j(x)$ and $T_j(x,y)$ are elements of $L(W,W) \oplus W$, and satisfy the Lie algebra relations given in section 3. Moreover, they satisfy Eq.(3.12b), i.e.

$$T_0(x, \overline{y \star z}) + T_1(z, \overline{x \star y}) + T_2(y, \overline{z \star x}) = 0.$$

Since the proof of this Theorem is lengthy, we will not go into it, (see [K-O.13]). Here,we simply mention that we utilized Eqs.(6.14), (6.25) and (6.26) for the purpose.

We also note that the choice of $\gamma_0 = \gamma_1 = \gamma_2 = 1$ requires $k^2 = -1$ and $\alpha\beta = -\frac{1}{2}$ by Eq.(7.36). Then the underlying field F must contain the square root of -1.

Let A be the normal triality algebra associated with a A-ternary algebra $(A^*, x \star y, xyz)$ (or a Kantor triple system (A^*, xyz)). Then from results of this section (the case of $\gamma_0 = \gamma_1 = \gamma_2 = 1$), we have an isomorphism of Lie algebras

$$L = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T(A,A) \cong L(W,W) \oplus W, \tag{7.38}$$

where $W = A^* \oplus A^*$.

According to [Kam.87], we recall the definition of Killing form $\gamma(x,y)$ of A^* ;

$$\gamma(x,y) = \frac{1}{2}\mathrm{Tr}(2R(x,y) + 2R(y,x) - L(x,y) - L(y,x)),$$

where $L(x,y)z = xyz,\ R(x,y)z = zxy,$ for any x, y and $z \in A^*$.

Then from straight forward calculations by using $2\alpha\beta = -1$, we obtain the following.

Proposition 7.7

Under the assumosion as in Eq.(7.38), for the Killing form $\gamma(x,y)$, we have

$$\gamma(x,y) = -\mathrm{Tr}\ (\mathrm{ad}\rho_1(x)\ \mathrm{ad}\rho_1(y)),$$
$$\gamma(\bar{x},\bar{y}) = -\mathrm{Tr}\ (\mathrm{ad}\rho_2(x)\ \mathrm{ad}\rho_2(y)).$$

8. BC_1-graded Lie Algebra of Type B_1

As we noted in the previous sections, any A-ternary algebra admitts two constructions of Lie algebras: The one given in section 3 exhibits the triality, but *not* the 5-graded structure, while the other one based upon the standard construction for (ε, δ)FKTS in section 7 manifests the explicit 5-graded nature but *not* the triality.

Here in this section, we will show that the second method can be used to prove that its associated Lie algebra L is a BC_1-graded Lie algebra of type B_1 ([Be-S.03]), i.e., there exists a sub-Lie algebra $sl(2)$ of L such that L regarded as the $sl(2)$ module is a direct sum of trivial, 3-dimensional, and 5-dimensional modules of $sl(2)$.

Let $(A^*,\ x \star y,\ xyz)$ be an A-ternary algebra with the unit element e for the structurable algebra $(A^*, x \star y)$ so that it satisfies

$$eex = x, \qquad 2xee + exe = 3x. \tag{8.1}$$

Following [K-O.10] or [E-K-O,13], let $R, M \in \text{End}\, A^*$ be defined by

$$Rx = xee, \qquad Mx = exe \tag{8.2}$$

so that Eq.(8.1) gives

$$M + 2R = 3id. \tag{8.3}$$

We then have, (assuming $2 \neq 0$, hereafter.)

Lemma 8.1

(1)

$$L(xye, e) = L(e, yxe) \tag{8.4a}$$

(2)

$$L(Rx, e) = L(e, Mx) \tag{8.4b}$$

(3)

$$L(Mx, e) = L(e, Rx) \tag{8.4c}$$

(4)

$$M^2 = R^2 \tag{8.4d}$$

(5)

$$K(x, e)e = (R - 1)x \tag{8.4e}$$

(6)

$$K(x,y) = \frac{1}{2}K(K(x,y)e,e). \tag{8.4f}$$

(7)

$$(R-1)(R-3) = 0 \tag{8.4g}$$

Proof

Setting $x = y = e$ we have

$$[L(u,v), L(e,e)] = L(uve,e) - L(e,vue).$$

Moreover, $eex = x$ implies

$$L(e,e) = \mathrm{id}\ (\equiv 1)$$

so that it yields Eq.(8.4a) by changing $u \to x$ and $v \to y$. Then Eqs.(8.4b) and (8.4c) are special cases of Eq.(8.4a) for either $x = e$ or $y = e$.

In order to prove Eq.(8.4d),Eq.(6.2) gives

$$xe(eee) = (xee)ee - e(exe)e + ee(xee)$$

which is rewritten as

$$Rx = R^2x - M^2x + Rx$$

i.e.$R^2 = M^2$. Moreover, Eq.(6.6) leads to $K(x,e)e = xee - eex = (R-1)x$ i.e. Eq.(8.4e). Similarly, we find

$$K(K(u,v)e,e) = L(e,e)K(u,v) + K(u,v)L(e,e) = 2K(u,v)$$

from Eq.(6.7) to give Eq.(8.4d). Finally, from Eqs.(8.4e) and (8.4d), we calculate

$$(R-1)x = K(x,e)e =$$
$$\frac{1}{2}K(K(x,e)e,e)e = \frac{1}{2}(R-1)K(x,e)e = \frac{1}{2}(R-1)^2x,$$

so that $R-1 = \frac{1}{2}(R-1)^2$ or $(R-1)(R-3) = 0$ as in Eq.(8.4e). Note that this relation is consistent with $M^2 = R^2$ in Eq.(8.3). □

In view of Eq.(8.4g), $(R-1)(R-3) = 0$, we can decompose A^* as in

$$A^* = V_1 \oplus V_3 \tag{8.5}$$

where we have set

$$V_1 = \{x|Rx = x,\ x \in A^*\} \tag{8.6a}$$

$$V_3 = \{x|Rx = 3x,\ x \in A^*\}. \tag{8.6b}$$

For the details of this decomposition, see also ([K-K.03]). Moreover by Eq.(6.11a), we have

$$\overline{x} = (2 - R)x, \tag{8.7}$$

so that we can rewrite Eqs.(8.6) also as

$$V_1 = \{x|\overline{x} = x, x \in A^*\}, \tag{8.8a}$$

$$V_3 = \{x|\overline{x} = -x, x \in A^*\}.$$

We next set

$$h = \begin{pmatrix} L(e,e), & 0 \\ 0, & -L(e,e) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.9a}$$

$$f = \begin{pmatrix} e \\ o \end{pmatrix}, \qquad g = \begin{pmatrix} 0 \\ e \end{pmatrix}. \tag{8.9b}$$

We see then $f, g \in W$ and $h \in L(W, W)$ so that they are elements of the Lie algebra $L(W, W) \oplus W$. Moreover, they satisfy the $sl(2)$ Lie relations of

$$[h, f] = f, \qquad [h, g] = -g, \qquad [f, g] = h \tag{8.10}$$

since we calculate

$$[h, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} = f$$

$$[h, g] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ -e \end{pmatrix} = -g$$

and

$$[f, g] = [\begin{pmatrix} e \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e \end{pmatrix}]$$

$$= \begin{pmatrix} L(e,e) - L(e,0), & K(e,0) \\ K(0,e), & -L(e,e) + L(0,e) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h,$$

by Eqs.(7.28) and (7.29).

We note moreover that we have

$$K(x,e)=0, \qquad \text{if } x\in V_1, \tag{8.11}$$

since we calculate

$$K(x,e)=l(x\star\overline{e}-e\star\overline{x})=l(x-\overline{x}).$$

Then, Eq.(8.4) implies

$$K(A^*,A^*)=K(V_3,e). \tag{8.12}$$

We can now construct 5-dimentional modules M_5 and 3-dimensional M_3 of the $sl(2)$ by .

$$M_5 = span$$

$$< \begin{pmatrix} 0, & 0 \\ K(x,e), & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} L(x,e), & 0 \\ 0, & -L(e,x) \end{pmatrix},$$
$$\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0, & K(x,e) \\ 0, & 0 \end{pmatrix} > \tag{8.13}$$

for $x\in V_3$ satisfying $\overline{x}=-x$, and

$$M_3 =$$

$$span < \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} L(x,e), & 0 \\ 0, & -L(e,x) \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix} > \tag{8.14}$$

for $x\in V_1$, satisfying $\overline{x}=x$. (Note $K(x,e)=0\ for\ x\in V_1$.)

We further note that

(1) If $\overline{x}=x$, then $Rx=x,\ Mx=x$, and $L(x,e)=L(e,x)$

(2) If $\overline{x}=-x$, then $Rx=3x,\ Mx=-3x$, and $L(x,e)=-L(e,x)$

by Eqs.(8.4b) and (8.4c), assuming the underlying field F to be of charach-teristic $\neq 3$ in addition, Especially, we need not consider elements of L of form

$$\begin{pmatrix} L(e,x), & 0 \\ 0, & -L(x,e) \end{pmatrix}.$$

Finally, the trivial modules can be constructed as follows.

For any $x, y \in A^*$, we set

$$u = \frac{1}{2}(R-1)(xye) \in V_3$$

$$v = -\frac{1}{2}(R-3)(xye) \in V_1. \tag{8.15}$$

We then have

$$X = \begin{pmatrix} L(u,e), & 0 \\ 0, & -L(e,u) \end{pmatrix} \in M_5 \tag{8.16a}$$

and

$$Y = \begin{pmatrix} L(v,e), & 0 \\ 0, & -L(e,v) \end{pmatrix} \in M_3. \tag{8.16b}$$

Then

$$\xi := \begin{pmatrix} L(x,y), & 0 \\ 0, & -L(y,x) \end{pmatrix} - \frac{1}{3}X - Y \tag{8.17}$$

can be verified to be elements of the trivial modules of the $sl(2)$. Then rewriting

$$\begin{pmatrix} L(x,y), & 0 \\ 0, & -L(y,x) \end{pmatrix} = \xi + \frac{1}{3}X + Y \in M_1 \oplus M_3 \oplus M_5$$

together with $K(A^*, A^*) = K(V_3, e)$, we have

$$M_1 := \{ \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix} | \phi(e) = \phi'(e) = 0,\ \phi, \phi' \in L(A^*, A^*)\},$$

and we obtain

$$L = L(W,W) \oplus W = M_1 \oplus M_3 \oplus M_5.$$

Therefore, we have found:

Proposition 8.2

Let $(A^*, x \star y, xyz)$ be an A-ternary algebra over the field F of charachteristic $\neq 2$, $\neq 3$. Then, its associated Lie algebra is a BC_1-graded Lie algebra of type B_1.

If we assume for simplicity the field F to be an algebraically closed field of charachteristic zero, then we can show conversely that any BC_1-graded Lie algebra of type B_1 can be constructed from some A-ternary algebra. For this purpose, it is convenient to use the terminology familiar in the angular momentum algebra in Quantum Mechanics (e.g.[C-D-L.77]) by setting

$$J_3 = h,\ J_+ = \sqrt{2}f,\ J_- = \sqrt{2}g \tag{8.18}$$

which satisfy

$$[J_3, J_\pm] = \pm J_\pm,\ [J_+, J_-] = 2J_3. \tag{8.19}$$

We write the generic irreducible state as

$$\Phi(j, m; \alpha) = |j, m; \alpha > \tag{8.20}$$

for $j = 0, 1, 2$, corresponding to the trivial, 3-dimensional, and 5-dimensional modules, while the sub-quantum number m can assume $2j + 1$ values of $j, j-1, \cdots, -(j-1), -j$. Also, α in Eq.(8.20) simply designates other labels. For example, we may identify $\alpha = x$, satisfying $\overline{x} = -x$ or $\overline{x} = +x$ for modules M_5 and M_3 in Eqs.(8.13) and (8.14). We note the commutation relations of

$$\begin{gathered}[J_3, \Phi(j, m; \alpha)] = m\Phi(j, m; \alpha),\\ [J_\pm, \Phi(j, m; \alpha)] = \pm\sqrt{(j \mp m)(j \pm m + 1)}\Phi(j, m \pm 1; \alpha).\end{gathered} \tag{8.21}$$

Now, the BC_1-graded Lie algebra of type B_1 is then 5-graded as in

$$L = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \tag{8.22}$$

when we set

$$g_m = \{x | J_3 x = mx, \quad (m = 0, \pm 1, \pm 2),\ x \in L\}. \tag{8.23}$$

Moreover, θ given by

$$\theta : \Phi(j, m; \alpha) \to (-1)^{j-m}\Phi(j, -m, \alpha) \tag{8.24}$$

is an automorphism of L of order 2, letting $g_m \leftrightarrow g_{-m}$. Especially, we note

$$\theta : h \to -h, \quad f \leftrightarrow g. \tag{8.25}$$

Then, by Theorem 4.1 of [E-K-O.13], $A = g_1$ becomes a Kantor triple system with respect to the triple product

$$xyz = [[x, \theta(y)], z] \tag{8.26}$$

for $x, y, z \in g_1$. Noting $f = \frac{1}{\sqrt{2}} J_+ \in g_1$, we calculate then

$$ffx = xff = fxf = x, \text{ for } x = \Phi(1, 1, ; \alpha) \in g_1$$

and

$$ffx = x, \ xff = -fxf = 3x \text{ for } x = \Phi(2, 1; \alpha) \in g_1,$$

so that they satisfy the condition of Eq.(6.10) for $e = f$. Therefore, by Theorem 6.1, $g_1 = A$ becomes a A-ternary algebra.

Remark 8.3

We can relax the condition in Proposition 8.2 and others as in [E-K-O.13], although we will not go into its detail. Also, an analogous theorem on Lie superalgebra associated with $(-1, -1)$ Freudenthal-Kantor triple system is given there. Similarly, some class of $(1, 1)$FKTS lead to BC_1-graded Lie algebra of type C_1 as well to a ternary system called a J-ternary algebra (see [E-O.11], and [A-B-G.02]).

In the previous sections and this section, we have studied relationship between the structurable algebra and the Kantor triple system. Since other (ε, δ) FKTS do not appear to have a direct connection to the triality relation, we did not discuss other (ε, δ) FKTS. Here, we simply mention that some $(-1, -1)$ FKTS have been used to construct exceptional Lie superalgebra $D(2, 1; \alpha)$, $G(3)$, and $F(4)$ (see [K-O.03], [E-K-O.03] and [E-K-O.05]). Also, some connections exist between $(1, 1)$ and $(-1, 1)$FKTS ([E-K-O.13]).

For other applications to mathematical physics with respect to triple systems and nonassociative algebras, we refer (for example, to see [O.95], [Sa.78]).

9 Triality group

The triality relations discussed in this note are of a local type. If $\sigma_j \in Epi(A)$ for $j = 0, 1, 2$ satisfy

$$\sigma_j(xy) = (\sigma_{j+1}x)(\sigma_{j+2}y), \tag{9.1}$$

in contrast to Eq.(1.1), it is called a global triality relation, and

$$G = \{\sigma_j | \sigma_j(xy) = (\sigma_{j+1}x)(\sigma_{j+2}y), \ \forall j = 0, 1, 2, \ \forall, x, y \in A\} \tag{9.2}$$

is a group (in general a Lie group) instead of a Lie algebra, which may be called triality group. Here, the indices over j are defined modulo 3. Its general structure is harder to analyze, and has not been studied much. Here, we will give a example based upon the symmetric composition algebra (see Example 2.2) satisfying

$$x(yx) = (xy)x = < x|x > y \tag{9.3}$$

where $< \cdot|\cdot >$ is a symmetric bi-linear non-degenerate form in A. For any two elements $a, b \in A$ satisfying $< a|a > = < b|b > = 1$, we set $a = a_1$, and $b = a_2$ and define a_3 by $a_3 = a_1 a_2$. We then find

$$a_j a_{j+1} = a_{j+2}, \qquad < a_j|a_j > = 1 \tag{9.4}$$

for $j = 1, 2, 3$, where the indices over j is defined again modulo 3, i.e. $a_{j\pm3} = a_j$. Introducing the multiplication operators $L(x)$ and $R(x)$ again by

$$L(x)y = xy, \qquad R(x)y = yx \tag{9.5}$$

we set

$$\sigma_j(a) = R(a_{j+1})R(a_{j+2}) \tag{9.6a}$$

$$\theta_j(a) = L(a_{j+2})L(a_{j+1}), \tag{9.6b}$$

for $j = 1, 2, 3$. We can then show the validity of

(i)

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y) \tag{9.7a}$$

(ii)

$$\theta_j(a)(xy) = (\theta_{j+1}(a)x)(\theta_{j+2}(a)y) \tag{9.7b}$$

(iii)

$$\sigma_j(a)\theta_j(a) = \theta_j(a)\sigma_j(a) = 1 \tag{9.7c}$$

(iv)

$$\theta_j(a)\theta_{j+1}(a)\theta_{j+2}(a) = \sigma_{j+2}(a)\sigma_{j+1}(a)\sigma_j(a) = 1 \tag{9.7d}$$

(v)

$$< \sigma_j(a)x|y > = < x|\theta_j(a)y > \tag{9.7e}$$

(vi)

$$< \sigma_j(a)x|\sigma_j(a)y >=< \theta_j(a)x|\theta_j(a)y >=< x|y > . \qquad (9.7f)$$

Especially, both $\sigma_j(a)$ and $\theta_j(a)$ satisfy the global triality relation. The details will be given elsewhere.

In final comments of this section, let

$$D = (D_1, D_2, D_3) \in s \circ Lrt(A), \quad i.e.,$$

$$D_j(xy) = (D_{j+1}x)y + x(D_{j+2}y). \ (j = 1, 2, 3). \qquad (9.8)$$

Moreover, suppose that the exponential map is *well-defined*, i.e.,

$$\xi_j(t) = exp(tD_j), \ \ (t \text{ is indeterminate variable and } t \in F) \qquad (9.9)$$

to be well-defined satisfying the Stone's theorem

$$\frac{d}{dt}\xi_j(t) = D_j\xi_j(t) = \xi_j(t)D_j. \qquad (9.10)$$

Then we have

Theorem 9.1

If $D \in s \circ Lrt(A)$, i.e., D_j satisfies Eq.(9.8), then $\xi_j(t)$ satisfies

$$\xi_j(t)(xy) = (\xi_{j+1}(t)x)(\xi_{j+2}(t)y) \ \ (x, y \in A). \qquad (9.11)$$

Conversely, the validity of Eq.(9.11) with Eq.(9.9) implies that of the local triality relation Eq.(9.8), i.e., we have the correspondence;

local triality $\leftrightarrow$ global triality.

Proof.

Set

$$F(t) = \xi_j(-t)\{(\xi_{j+1}(t)x)(\xi_{j+2}(t)y)\}. \qquad (9.12)$$

Then, we calcultate

$$\frac{d}{dt}F(t) = \xi_j(-t)\{-D_j[(\xi_{j+1}(t)x)(\xi_{j+2}(t)y)]$$

$$+(D_{j+1}\xi_{j+1}(t)x)(\xi_{j+2}(t)y) + (\xi_{j+1}(t)x)(D_{j+2}\xi_{j+2}(t)y))\}. \qquad (9.13)$$

Thereofre, if Eq.(9.8) holds for any $x \in A$, then we have

$$\frac{d}{dt}F(t) = 0 \tag{9.14}$$

by replacing $\xi_{j+1}(t)x \to x$ and $\xi_{j+2}(t)y \to y$. Then, $F(t)$ is independent for the value of t, so that

$$F(t) = F(0) = xy$$

by Eq.(9.12). Noting $\xi_j(t)\ \xi_j(-t)$ =identity, we have the fact that Eq.(9.12) gives Eq.(9.11).

Conversely if Eq.(9.11) holds, then Eq.(9.12) implies $F(t) = xy$, since $\xi_j(-t)\xi_j(t) = 1$. Then Eqs.(9.13) and (9.14) for $t = 0$ yields.

$$D_j(xy) = (D_{j+1}x)y + x(D_{j+2}y).\ \ i.e.\ \ Eq.(9.8).$$

This completes the proof. □

Remark 9.2

The well definedness of the exponential map is $o.k$, if A is a finite-dimensional algebra over the real or complex field F, since

$$\mathrm{cxp}(tD_j) = \sum_{n=0}^{\infty} \frac{1}{n!}(tD_j)^n$$

is convergent in some suitably chosen topology.

For the details of triality groups, we will discuss in forcecoming paper.

Appendux (N.Kamiya and S.Okubo)

$L(A) =$

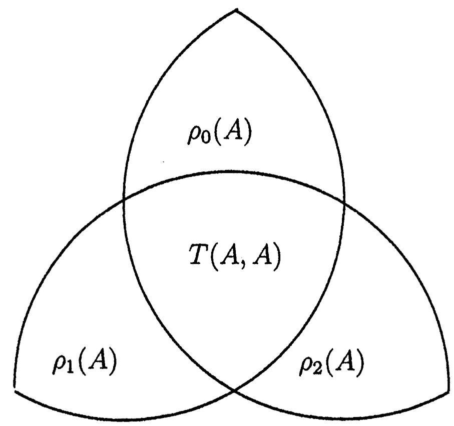

Fig.1 *Graphical Representation of the Lie Algebra* $L(A)$.

$V =$

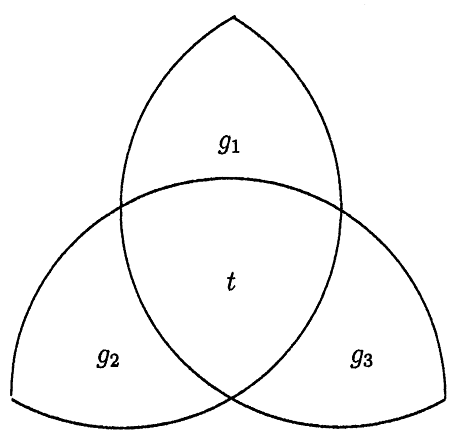

Fig.3 *Graphical Representation of* V.

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