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MATRIX CONTINUED FRACTIONS AND THE EXPANSIONS OF SOME FUNCTIONS OF BESSEL

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Abstract

In this paper we recall some results of matrix functions with real coefficients. The aim of this paper is to provide some properties and results of continued fractions with matrix arguments. Then we give continued fractions expansions of Bessel functions $J_0(A)$ and $J_1(A)$ where A is matrix.

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1 Introduction and motivation

Using Padé approximants, equivalents to continued fraction approximants of the matrix exponential, Varga [10] studied and analyzed certain discrete approximations to solutions of self-adjoint parabolic differential equations. The matrix continued fraction used by Varga is an example of noncommutative continued fractions about which not much is known. Thus, the theory of continued fractions whose elements are operators have many applications in various areas of mathematics and allow an acceleration of the convergence of a number of approximations. In this direction, and generally in a Banach space, few convergence results on non-commutative continued fractions are known. Two theorems are cited by Wynn [11], where he gives some aspects for the theory of continued fractions which the elements do not commute. In Banach spaces, generalizations of some results of real cases are published by Haydan [2] and Negoescu [8]. In [9], note that we gave a criterion provides a necessary condition for the convergence of the noncommutative matrix continued fraction of the form $K(I/A_n)$, that we have generalized to the case of $K(B_n/A_n)$. We also expressed a sufficient condition for the convergence of $K(B_n/A_n)$.

In this work, we quote some early results that link the elements, the numerators and denominators of convergent continued fractions matrices. Then, We recall some fundamental relations of these fractions we will need later. Towards the end, we give the continued fractions expansions of certain Bessel functions $J_0(A)$ and $J_1(A)$ of a matrix A after proving the convergence of these developments.

2 Definitions and notations

2.1 Generalities

Throughout, we denote by $\mathcal{M}_m$ the set of $m \times m$ real (complex) matrices endowed with the classical norm defined by

$$\forall A \in \mathcal{M}_m, \quad ||A|| = \sup_{x\neq 0}\{\frac{||Ax||}{||x||}\} = \sup\{||Ax||, ||x|| = 1\}. \qquad (1)$$

This norm satisfies the inequality $||AB|| \leq ||A||\,||B||$.

Let $A \in \mathcal{M}_m$, A is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$\forall x \in R^m,\ < Ax, x > \geq 0$ (resp. $\forall x \in\ R^m,\ x \neq 0,\ < Ax, x >> 0$)

where $< .,. >$ denotes the standard scalar product of R^m defined by x=($x_1, ..., x_m) \in R^{\mathbf{m}}$, $y = (y_1, ..., y_m) \in R^m :< x, y >= \sum_{i=1}^m x_i y_i$.

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetrics, we write $A \leq B$ if $B - A$ is positive semidefinite. Henceforth, whenever we say that $A \in \mathcal{M}_m$ is positive semidefinite (or positivedefinite), it wille be assumed that A is symmetric. It is easy to see that if $A \leq B$ then $CAC \leq CBC$ for all symmetric matrix C.

For any $A, B \in \mathcal{M}_m$ with B invertible, we write $\frac{A}{B} = B^{-1}A$. In particular, if $A = I$, where I is the m^{th} order identity matrix, then we write $\frac{I}{B} = B^{-1}$. It is clear that for any invertible matrix C, we have

$$\frac{CA}{CB} = \frac{A}{B} \neq \frac{AC}{BC}. \tag{2}$$

Definition 2.1 *Let $(A_n)_{n\geq 0}, (B_n)_{n\geq 1}$ be two nonzero sequences of $\mathcal{M}_m$. The continued fraction of (A_n) and (B_n) denoted by $K(B_n/A_n)$ is the quantity*

$$A_0 + \frac{B_1}{A_1 + \frac{B_2}{A_2 + \ldots}} = \left[A_0; \frac{B_1}{A_1}, \frac{B_2}{A_2}, \ldots\right].$$

Sometimes, we use briefly the notation

$$\left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty}. \tag{3}$$

The fractions $\frac{B_n}{A_n}$ and $\frac{p_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n}$ are called, respectively, the n^{th} partial quotient and the n^{th} convergent of the continued fraction $K(B_n/A_n)$.
When $B_n = I$ for all $n \geq 1$, then $K(I/A_n)$ is called an ordinary continued fraction.

Proposition 2.2 *The elements $(P_n)_{n\geq -1}$ and $(Q_n)_{n\geq -1}$ of the n^{th} convergent of $K(B_n/A_n)$ are given by the relationships*

$$\begin{cases} P_{-1} = I, & P_0 = A_0 \\ Q_{-1} = 0, & Q_0 = I \end{cases} \ \text{and} \ \begin{cases} P_n = A_n P_{n-1} + B_n P_{n-2} \\ Q_n = A_n Q_{n-1} + B_n Q_{n-2} \end{cases}, \quad n \geq 1. \tag{4}$$

Proof. We prove it by induction. The proof of the next proposition is elementary and we left it to the reader.

Proposition 2.3 *For any two matrices C and D with C invertible, we have*

$$C\left[A_0, \frac{B_k}{A_k}\right]_{k=1}^{n} D = \left[CA_0D; \frac{B_1D}{A_1C^{-1}}, \frac{B_2C^{-1}}{A_2}, \frac{B_k}{A_k}\right]_{k=3}^{n}. \tag{5}$$

2.2 Definition of convergent matrix continued fraction

The continued fraction$K(B_n/A_n)$ converges in $\mathcal{M}_m$ if the sequence $(F_n) = (\frac{P_n}{Q_n}) = (Q_n^{-1}P_n)$ converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that $\lim_{n\to+\infty} ||F_n - F|| = 0$. In the other case, we say that $K(B_n/A_n)$ is divergent. It is clear that

$$\frac{P_n}{Q_n} = A_0 + \sum_{i=1}^{n}(\frac{P_i}{Q_i} - \frac{P_{i-1}}{Q_{i-1}}). \tag{6}$$

From (5), we see that the continued fraction $K(B_n/A_n)$ converges in $\mathcal{M}_m$ if and only if the series $\sum_{n=1}^{+\infty}(\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}})$ converge in $\mathcal{M}_m$.

3 Convergence of continued fraction of the form $K(B_n/A_n)$

3.1 Equivalent continued fractions

Definition 3.4 *Let $(A_n), (B_n)$ (C_n) and (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1 \geq$, where F_n and G_n are the n^{th} convergent of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.*

Now, we will present two lemmas concerning the real continued fractions which are importants in the sequel.

Lemma 3.5 *([4]) Let (r_n) be a non-zero sequence of real numbers. The continued fractions*

$$\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots\right] \quad and \quad \left[a_0; \frac{r_1 b_1}{r_1 a_1}, \frac{r_2 r_1 b_2}{r_2 a_2}, \frac{r_3 r_2 b_3}{r_3 a_3}, \ldots\right] \tag{7}$$

are equivalent.

In the following lemma, from the development of a function given by the Taylor series, we give the development in continued fractions of the series that was established by Euler.

Lemma 3.6 *([5]) Let f be a function with the Taylor serie development is $f(x) = \sum_{n=0}^{+\infty} c_n x^n$ in $D \subset R$. Then, the development in continued fraction of $f(x)$ is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty} c_n x^n \\ &= \left[c_0; \frac{c_1 x}{1}, \frac{-c_2 x}{c_1 + c_2 x}, \frac{-c_1 c_3 x}{c_2 + c_3 x}, \ldots, \frac{-c_{n-2} c_n x}{c_{n-1} + c_n x}, \ldots\right] \\ &= \left[\frac{c_0}{1}, \frac{-c_1 x}{c_0 + c_1 x}, \frac{-c_0 c_2 x}{c_1 + c_2 x}, \frac{-c_1 c_3 x}{c_2 + c_3 x}, \ldots, \frac{-c_{n-2} c_n x}{c_{n-1} + c_n x}, \ldots\right]. \end{aligned}$$

Remark 3.7 *Let (A_n) and (B_n) be two sequences of $\mathcal{M}_m$, we notice that we can write the first convergents of the continued $K(B_n/A_n)$ by:*
$F_1 = A_0 + A_1^{-1} B_1 = A_0 + (B_1^{-1} A_1)^{-1}$,

$$\begin{aligned} F_2 &= A_0 + \left(A_1 + A_2^{-1} B_2\right)^{-1} B_1 \\ &= A_0 + \left(B_1^{-1} A_1 + (B_2^{-1} A_2 B_1^{-1})^{-1}\right)^{-1}. \end{aligned}$$

If we put, $A_1^* = B_1^{-1} A_1$ and $A_2^* = B_2^{-1} A_2 B_1$, we have

$$F_1 = A_0 + \frac{I}{A_1^*}, \quad F_2 = A_0 + \frac{I}{A_1^* +}\frac{I}{A_2^*}. \tag{8}$$

Generally, we prove by a recurrence that if we put for all $k \geq 1$,

$$A_{2k}^* = (B_{2k} \ldots B_2)^{-1} A_{2k} B_{2k-1} \ldots B_1 \quad and \quad A_{2k+1}^* = (B_{2k+1} \ldots B_1)^{-1} A_{2k+1} B_{2k} \ldots B_2, \tag{9}$$

then the continued fractions $A_0+K(B_n/A_n)$ *and* $A_0+K(I/A_n^*)$ *are equivalent.*

Il follows that the convergence of one of these continued fractions implies the convergence of the other continued fraction.

3.2 Convergence criteria of $K(B_n/A_n)$

Theorem 3.8 *([9]) Let* $(A_n)_{n\geq 0}$ *be a sequence of* $\mathcal{M}_m$. *If the continued fraction* $K(I/A_n)$ *converges in* $\mathcal{M}_m$, *then the series* $\sum_{n=0}^{\infty} ||A_n||$ *diverges.*

From theorem 3.7, we deduce the following corollary.

Corollary 3.9 *If the series* $\sum_{n=0}^{\infty} ||A_n||$ *converges absolutely in* $\mathcal{M}_m$, *then the ordinary continued fraction* $K(I/A_n)$ *diverges in* $\mathcal{M}_m$.

We recall here a criteria that gives a sufficient condition for the convergence of continued fraction in $\mathcal{M}_m$.

Theorem 3.10 *([9]) Let* $(A_n), (B_n)$ *be two sequences of* $\mathcal{M}_m$. *If*

$$||(B_{2k-2}...B_2)^{-1}\, A_{2k-1}^{-1}\, B_{2k-1}...B_1|| \leq \alpha \text{ and } ||(B_{2k-1}...B_1)^{-1}\, A_{2k}^{-1}\, B_{2k}...B_2)|| \leq \beta \tag{10}$$

for all $k \geq 1$, *where* $0 < \alpha < 1$, $0 < \beta < 1$ *and* $\alpha\beta \leq 1/4$, *then the continued fraction* $K(B_n/A_n)$ *converges in* $\mathcal{M}_m$.

4 Main results

4.1 Some reminders about Bessel functions

Bessel functions occupy a very important place in the problems solutions with cylindrical symmetry. We can say that they are important in the hierarchy of functions which are necessaries to tabulate. Bessel functions come immediately after the trigonometric functions, logarithm and exponential function but it is customary to fit them in the class of so-called special functions. There are many books deal with remarkable properties of these functions. Our goal is to show how to write the development of continued fraction of Bessel functions $J_0(x)$ and $J_1(x)$ and to study the convergence of these functions.

The following differential equation

$$x^2\frac{d^2F}{dx^2}(x) + x\frac{dF}{dx}(x) + \left(x^2 - v^2\right)F(x) = 0$$

is called Bessel differential equation of order v where v is a real number. The solution of this equation can be written by

$$F(x) = C_1 J_v(x) + C_2 Y_v(x)$$

where J_v and Y_v are Bessel functions of order v of the first and second kinds respectively which are given by:

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k}}{k!\Gamma(v+k+1)}, \qquad |\arg x| < \pi, \quad v \in C$$

where Γ is gamma function and

$$Y_v(x) = \frac{J_v(x)\cos(\pi v) - J_{-v}(x)}{\sin(\pi v)}, \qquad |\arg x| < \pi, \quad v \in C \setminus Z.$$

Let $n \in N$, we define the Bessel function J_n of index n. The power series of J_n is

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k}.$$

4.2 Continued fractions expansions of $J_0(A)$ and $J_1(A)$

4.2.1 Convergence of $J_0(A)$

Let us consider $x \in R$, the power series of $J_0(x)$ can be written by

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{2k}}{(k!)^2}.$$

We search the development of continued fraction of $J_0(x)$ which is in the form $K(b_n/a_n)$.

Lemma 4.11 *The continued fraction expansion of $J_0(x)$ is given by*

$$J_0(x) = \left[1; \frac{\frac{-x}{4}}{1}, \frac{x^2}{4^2 - x^2}, \frac{(-1)^n .x^{n-1}}{4^{n-2}((n-2)!)^2 (4n^2 - x^2)}\right]_{n=3}^{+\infty}.$$

Proof. We use Lemma 3.6 for the function:
$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{2k}}{(k!)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^k}{4^k (k!)^2} x^k$, by putting $C_n = \frac{(-1)^n}{(n!)^2 4^n} x^n$.
For $n \geq 3$, we get that :

$$\begin{aligned} C_{n-2}.C_n &= \frac{(-1)^{n-2}}{((n-2)!)^2 4^{n-2}} x^{n-2}. \frac{(-1)^n}{(n!)^2 4^n} x^n \\ &= \frac{(-1)^{2n-2} x^{2n-2}}{(n!)^2 ((n-2)!)^2 4^{2n-2}} \end{aligned}$$

Furthermore, we have

$$\begin{aligned} C_{n-1} + C_n x &= \frac{(-1)^{n-1} x^{n-1}}{((n-1)!)^2 4^{n-1}} + \frac{(-1)^n .x^{n+1}}{(n!)^2 4^n} \\ &= \frac{(-1)^{n-1} x^{n-1} \left[4(n!)^2 - ((n-1)!)^2 .x^2\right]}{((n-1)!n!)^2 4^n} \\ &= \frac{(-1)^{n-1} (4n^2 - x^2)}{4^n (n!)^2} x^{n-1} \end{aligned}$$

As a result,

$$\begin{aligned} \frac{C_{n-2} C_n}{C_{n-1} + C_n x} &= \frac{(-1)^{n-2} x^{2n-2}}{(n!(n-2)!)^2 4^{2n-2}}. \frac{4^n (n!)^2}{(-1)^{n-1} (4n^2 - x^2) .x^{n-1}}, \quad n \geq 3 \\ &= \frac{(-1)^{n-1} x^{n-1}}{4^{n-2} ((n-2)!)^2 (4n^2 - x^2)}. \end{aligned}$$

In particular, we have

$$C_0 = 1, \quad C_1 = \frac{-x}{4}, \quad C_2 = \frac{1}{4^3}x^2, \quad \frac{C_2 x}{C_1 + C_2 x} = \frac{x^2}{x^2 - 4^2}.$$

Therefore, the development of continued fraction of $J_0(x)$ is

$$J_0(x) = \left[1; \frac{\frac{-x}{4}}{1}, \frac{x^2}{4^2 - x^2}, \frac{-x^2}{4^1(43^2 - x^2)}, \frac{(-1)^n x^{n-1}}{4^{n-2}((n-2)!)^2(4n^2 - x^2)}\right]_{n=4}^{\infty}.$$

Now, we treat the matrix case of $J_0(x)$.

Theorem 4.12 *Let A be a matrix of $\mathcal{M}_m$, such that $\|A\| = \alpha$ where $\alpha \in R$ and $\alpha < \frac{1}{2}$. The continued fraction*

$$\left[I; \frac{\frac{-A}{4}}{I}, \frac{A^2}{4^2 I - A^2}, \frac{(-1)^n A^{n-1}}{4^{n-2}((n-2)!)^2(4n^2 I - A^2)}\right]_{n=3}^{+\infty}$$

converges in $\mathcal{M}_m$. Furthermore, the previous continued fraction is the continued fraction of $J_0(A)$ so

$$J_0(A) = \left[I; \frac{\frac{-A}{4}}{I}, \frac{A^2}{4^2 I - A^2}, \frac{(-1)^n A^{n-1}}{4^{n-2}((n-2)!)^2(4n^2 I - A^2)}\right]_{n=3}^{+\infty}.$$

Before proving this theorem we need to present the following proposition:

Proposition 4.13 *([7]) Let $C \in \mathcal{M}_m$ such that $\|C\| < 1$, then the matrix $I - C$ is invertible and we have*

$$\left\|(I - C)^{-1}\right\| \leq \frac{1}{1 - \|C\|}.$$

Proof of theorem 4.12 We study the convergence of the continued fraction $A_0 + K(B_k/A_k)$ with

$$\begin{cases} A_0 &= I, \quad A_1 = I, \quad A_2 = 4^2 I - A^2, \\ B_1 &= -A/4, \quad B_2 = A^2 \end{cases}$$

$$\text{and} \quad \begin{cases} B_k &= (-1)^k A^{k-1} \\ A_k &= 4^{k-2}\left((k-2)!\right)^2 \left(4k^2 I - A^2\right) \end{cases} \quad \text{for } k \geq 3.$$

Check that the conditions of theorem 3.10 are satisfied. We have

$$B_{2k-2} \ldots B_2 = \pm A^{(2k-3)+(2k-5)+\ldots+3+2},$$

$$A_{2k-1}^{-1} = \frac{1}{4^{2k-3}\left((2k-3)!\right)^2}\left(4(2k-1)^2 I - A^2\right)^{-1} \quad \text{and}$$

$$B_{2k-1}B_{2k-3}\ldots B_1 = \frac{1}{4}A^{(2k-2)+(2k-4)+\cdots 2+1}.$$

Then,

$$\begin{aligned} \left\|\left(B_{2k-2}\ldots B_2\right)^{-1} A_{2k-1}^{-1} B_{2k-1}\ldots B_1\right\| &= \left\|\left(A^{(2k-3)+(2k-5)+\ldots+3+2}\right)^{-1} \frac{1}{4^{2k-3}\left((2k-3)!\right)^2} \times \right. \\ &\quad \left. \left(4(2k-1)^2 I - A^2\right)^{-1}\frac{1}{4}.A^{(2k-2)+(2k-4)+\cdots 2+1}\right\|. \\ &\leq \frac{1}{4^{2k-2}\left((2k-3)!\right)^2}\left\|A^{-[(2k-3)+(2k-5)+\ldots+3+2]} \times \right. \\ &\quad \left. \left(4(2k-1)^2 I - A^2\right)^{-1} A^{(2k-2)+\ldots+2+1}\right\|. \end{aligned}$$

Now, the matrices $\left(4(2k-1)^2 I - A^2\right)^{-1}$ and $A^{(2k-2)+(2k-4)+\ldots+2+1}$ commute, so the above inequality becomes

$$\begin{aligned} \left\|\left(B_{2k-2}\cdot\cdot B_2\right)^{-1} A_{2k-1}^{-1} B_{2k-1}\cdot\cdot B_1\right\| &\leq \frac{1}{4^{2k-1}}\frac{1}{\left((2k-3)!\right)^2}\frac{1}{(2k-1)^2} \times \\ &\quad \left\|\left(I - \frac{A^2}{4(2k-1)^2}\right)^{-1}\frac{A^{2k-2}\ldots A^2 A}{A^{2k-3}\ldots A^3 A^2}\right\|. \end{aligned}$$

According to proposition (4.13) and the fact that $\|A\| < 1$, we obtain

$$\left\|\left(I - \left(\frac{A}{2(2k-1)}\right)^2\right)^{-1}\right\| \leq \frac{1}{1 - \left\|\frac{A}{2(2k-1)}\right\|^2} < 1.$$

It implies that for k large enough, we get

$$\left\|\left(B_{2k-2}\ldots B_2\right)^{-1} A_{2k-1}^{-1} B_{2k-1}\ldots B_1\right\| < \|A\| = \alpha < \frac{1}{2}.$$

To prove the second inequality of theorem (3.10), we have

$$\begin{aligned}\left(B_{2k-1}...B_1\right)^{-1}\ A_{2k}^{-1}\ B_{2k}...B_2 &= \pm 4\frac{\left(A^{(2k-2)+...2+1}\right)^{-1}}{4^{2k-2}\left((2k-2)!\right)^2}\left(16k^2I-A^2\right)^{-1}\times \\ &\quad A^{(2k-1)+....+3+2} \\ &= \pm\frac{\left(A^{(2k-2)+...2+1}\right)^{-1}}{4^{2k-1}\left((2k-2)!\right)^2 k^2}\left(I-\left(\frac{A}{16k}\right)^2\right)^{-1}\times \\ &\quad A^{(2k-1)+....+3+2}\end{aligned}$$

Again using the fact that the matrices $\left(16k^2I-A^2\right)^{-1}$, $A^{(2k-1)+...+3+2}$ commute, the proposition (4.13) and passing to the norm, we get

$$\left\|\left(B_{2k-1}...B_1\right)^{-1}\ A_{2k}^{-1}\ B_{2k}...B_2\right\| \leq \frac{1}{4^{2k-1}.\left((2k-2)!\right)^2.k^2}\left\|A\right\| < \alpha < \frac{1}{2}$$

which completes the proof.

4.2.2 Convergence of $J_1(A)$

Let x be a real number, the power series expansion of $J_1(x)$ is

$$J_1(x) = \sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{x}{2}\right)^{2k+1}}{k!\,(k+1)!}.$$

In the next lemma, we give its continued fraction expansion.

Lemma 4.14 *The continued fraction expansion of $J_1(x)$ is given by*

$$J_1(x) = \left[\frac{x}{2};\frac{\frac{-x^3}{2^4}}{1},\frac{x^2}{\left(2^3\cdot 3!-x^2\right)},\frac{(-1)^n x^{n+1}}{2^{2n-3}(n-2)!\,(n-1)!\,(4n(n+1)-x^2)}\right]_{n=3}^{+\infty}.$$

Proof. We have

$$J_1(x) = \sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{x}{2}\right)^{2k+1}}{k!\,(k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{2^{2k+1}k!(k+1)!}x^k,$$

taking $C_n = (-1)^n \frac{1}{2^{2n+1}} \cdot \frac{x^{n+1}}{n!(n+1)!}$ for $n \geq 1$,we find

$$C_{n-2}C_n x = \frac{x^{2n+1}}{2^{4n-2}(n-2)!(n-1)!n!(n+1)!} \quad \text{for all } n \geq 3.$$

Furthermore, we obtain

$$C_{n-1} + C_n x = \frac{(-1)^{n-1}.x^n}{2^{2n-1}(n-1)!n!} \cdot \frac{4n(n+1) - x^2}{2^2 n(n+1)}.$$

As a result,

$$\frac{C_{n-2}C_n x}{C_{n-1} + C_n x} = \frac{(-1)^{n-1}.x^{n+1}}{2^{2n-3}(n-2)!(n-1)!(4n(n+1) - x^2)}.$$

In particular case, we have

$$C_0 = \frac{x}{2}, \quad C_1 = \frac{-x^2}{2^4}, \quad C_2 = \frac{x^3}{2^6.3!}, \quad \frac{-C_2 x}{C_1 + C_2 x} = \frac{x^2}{(2^2.3! - x^2)}.$$

Hence, the continued fraction expansion of $J_1(x)$ is

$$J_1(x) = \left[\frac{x}{2}; \frac{\frac{-x^3}{2^4}}{1}, \frac{x^2}{(2^2.3! - x^2)}, \frac{(-1)^n x^{n+1}}{2^{2n-3}(n-2)!(n-1)!(4n(n+1) - x^2)}\right]_{n=3}^{+\infty}.$$

Then, we treat the matrix case.

Theorem 4.15 *Let A be a matrix of $\mathcal{M}_m$ such that $\|A\| = \alpha$ where $\alpha \in R$ and $\alpha < \frac{1}{2}$. The continued fraction*

$$\left[\frac{A}{2}; \frac{\frac{-A^2}{2}}{1}, \frac{A^2}{(2^2 \cdot 3!I - A^2)}, \frac{(-1)^n.A^{n+1}}{2^{2n-3}(n-2)!(n-1)!(4n(n+1)I - A^2)}\right]_{n=3}^{+\infty}$$

converge in $\mathcal{M}_m$. Furthermore, the previous continued fraction is the continued fraction expansion of $J_1(A)$. Hence,

$$J_1(A) = \left[\frac{A}{2}; \frac{\frac{-A^3}{2^4}}{1}, \frac{A^2}{(2^2.3!I - A^2)}, \frac{(-1)^n A^{n+1}}{2^{2n-3}(n-2)!(n-1)!(4n(n+1)I - A^2)}\right]_{n=3}^{+\infty}.$$

Proof. We study the convergence of the continued fraction $A_0+K\left(B_k/A_k\right)$ with :

$$\begin{cases} A_0 &= I, \quad A_1 = I, \quad A_2 = 2^4\left(2^2.3!I - A^2\right), \\ B_1 &= -A^2/4, \quad B_2 = A^2, \end{cases}$$

and for $k \geq 3$,

$$\begin{cases} B_k &= (-1)^{k-1} A^{k+1}, \\ A_k &= 2^{2k-3}(k-2)!\,(k-1)!\,(4k(k+1))\,I - A^2) \end{cases}$$

By the same manipulation as in $J_0(A)$, we show that the condition of Theorem 3.10 are satisfied.

$$B_{2k-2}.....B_2 = \pm A^{(2k-1)+(2k-3)+......+5+2},$$

$$A_{2k-1}^{-1} = \frac{1}{2^{2k-4}(2k-3)!\,(2k-2)!2k\,(8k-4)}\left(I - \frac{A^2}{2k\,(8k-4)}\right)^{-1},$$

then

$$\left\|\left((B_{2k-2}...B_2)^{-1}\right)A_{2k-1}^{-1}B_{2k-1}...B_1\right\| = \left\|\frac{A^{-((2k-1)+......+5+2)}}{2^{2k-4}(2k-3)!\,(2k-2)!2k\,(8k-4)}\times \left(I - \frac{A^2}{2k\,(8k-4)}\right)^{-1}\frac{1}{4}A^{2k+(2k-2)+......+4+2}\right\|.$$

It follows that

$$\left\|\left((B_{2k-2}...B_2)^{-1}\right)A_{2k-1}^{-1}B_{2k-1}...B_1\right\| \leq \frac{1}{2^{2k-3}(2k-3)!\,(2k-2)!2k\,(8k-4)}\times \left\|\left(I - \frac{A^2}{2k\,(8k-4)}\right)^{-1}\frac{A^{2k}A^{2k-2}....A^4A^2}{A^{2k-1}A^{2k-3}......A^5A^2}\right\|.$$

Since $\|A\| < 1$, we get

$$\left\|\frac{A^{2k}A^{2k-2}....A^4A^2}{A^{2k-1}A^{2k-3}....A^5A^2}\right\| \leq \left\|A^{k-1}\right\| \leq \|A\|^{k-1} \leq \|A\|.$$

On the other side, when k is large, we have

$$\frac{1}{2^{2k-4}(2k-3)!(2k-2)!2k(8k-4)} < 1.$$

By proposition 3.14, we finally obtain

$$\left\| \left(I - \frac{A^2}{2k(8k-4)} \right)^{-1} \right\| \leq \frac{1}{1 - \left\| \frac{A}{\sqrt{2k(8k-4)}} \right\|^2} < 1.$$

Which implies that

$$\left\| \left((B_{2k-2} ... B_2)^{-1} \right) A_{2k-1}^{-1} B_{2k-1} ... B_1 \right\| \leq |A| = \alpha < \frac{1}{2}.$$

The proof of the second inequality of Theorem 3.10 is similar to the first one.

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