ON FRACTIONAL ELZAKI TRANSFORM AND MITTAG – LEFFLER FUNCTION

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Abstract

In this paper Elzaki transform of fractional order through the Mittag - Leffler function is introduced. Employing the same, few properties of Elzaki transform and its inversion are obtained.

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1. Introduction

Numerous integral transforms, namely, Fourier, Laplace, etc. are used to solve the differential and integral equations. Elzaki transform is introduced, see Elzaki [4]. Elzaki, et al. [5, 6, 7] describe the Elzaki transform as rivals of the Sumudu transform in solving problems related to telegraph equation, and integral and differential equations. More of its properties are proved in [5, 6, 7, 8, 9]

Elzaki transform is extended to distribution spaces and Boehmian spaces [1, 2] , and the same is proved for distribution spaces and to obtain solution of Abel integral equation [15]. Fractional Natural transform using Mittag - Leffler function, is obtained and its properties are discussed in [14]. In this paper, we introduce the Elzaki transform of fractional order using Mittag - Leffler function and study its properties.

In what follows, are the definition of the Elzaki transform and its properties. Consider a set A of function $f(t)$ of exponential order, which is defined by

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{|t|/k_j}; t \in (-1)^j \times [0, \infty)\},$$

where M is a constant of finite number and k_1 and k_2 may be finite or infinite.

Elzaki transform is given by

$$\mathbb{E}[f(t)] = T(v) = v \int_0^{\infty} e^{-(\frac{t}{v})} f(t) dt \quad , \quad t \geq 0, k_1 \leq v \leq k_2 \qquad (1)$$

The **duality relation between the Elzaki transform and the Laplace transform** is suggested by

$$T(v) = vL(1/v) \quad ; \quad L(s) = sT(1/s) \qquad (2)$$

where L denotes the Laplace transform and T is the Elzaki transform.

The **properties of Elzaki transform** may be enumerated as

1. The Elzaki transform of *derivative* of $f(t)$ and *nth order derivative* of $f(t)$ are, respectively, defined by

$$\mathbb{E}[f'(t)] = T'(v) = \frac{T(v)}{v} - vf(0) \qquad (3)$$

and

$$\mathbb{E}[f^{(n)}(t)] = T^{(n)}(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0) \quad . \tag{4}$$

2. When $f(t) = t^n$,the Elzaki transform reduces to

$$\mathbb{E}[t^n] = n! v^{n+2} \tag{5}$$
$$= \Gamma(n+1) v^{n+2} \; . \tag{6}$$

3. If $F(v)$ and $G(v)$ are Elzaki transforms of the functions $f(t)$ and $g(t)$, then the *convolution* is given by

$$\mathbb{E}[(f * g)(t)] = \frac{1}{v} \; F(v)G(v) \quad . \tag{7}$$

Theorem 1 [Inversion formula of Elzaki transform] [21] : Let $E(v)$ be the Elzaki transform of $f(t)$ such that

(i) $sE(1/s)$ is a meromorphic function, with singulartities having $\mathrm{Re}(s) < \alpha$, and

(ii) there exists a circular region Γ with radius R , positive constants M and K, with

$$|sE(1/s)| < M/R^K \tag{8}$$

and

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{st} s\mathbb{E}(1/s) ds \quad .$$

i.e.

$$f(t) = \sum \mathrm{Res} \; [e^{st} s\mathbb{E}(1/s)] \quad . \tag{9}$$

Further, using (2), Equation (11) can also be written as

$$f(t) = \sum \mathrm{Res} \; [\frac{1}{v} e^{t/v} \mathbb{E}(v)] \quad . \tag{10}$$

The Mittag – Leffler function [17] is a function, which is a direct generalization of the exponential function, and has an affinity for fractional calculus.

One parameter representation of the Mittag – Leffler function is given by [16]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k + 1)} \quad , \alpha > 0 \tag{11}$$

Whereas two parameter Mittag – Leffler function [3] is represented as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k + \beta)} \quad , \alpha, \beta, z \in \mathbb{C}, \mathrm{Re}(\alpha, \beta) > 0 \tag{12}$$

with $\mathbb{C}$ being the set of complex numbers.

Special cases of the Mittag – Leffler function are

(i) $E_\alpha(z) = \frac{1}{1-z} \quad , |z| < 1$

(ii) $E_1(z) = e^z$

(iii) $E_2(z) = \cosh(\sqrt{z}), z \in \mathbb{C}$

(iv) $E_{1,1}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum\limits_{k=0}^{\infty} \frac{z^k}{(k)!} = e^z$

(v) $E_{1,2}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z} \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \frac{e^z - 1}{z}$.

Following relations related to the Mittag – Leffler function may be useful.

(i) $\frac{d^m}{dz^m} E_m(z^m) = E_m(z^m)$

(ii) $E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}$

(iii) $\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta - 1) E_{\alpha,\beta}(z)}{\alpha z}$

Different techniques are employed to solve fractional differential equations [2, 16, 17, 18, 19, 20]. Using the Mittag – Leffler function and its properties, different types of integral transforms and some functions are defined and studied by the researchers [12, 13, 14, 15, 16, 17, 18].

In the following section, we give a brief introduction to fractional derivative that is employed, which is followed by the Elzaki transform of fractional order via the Mittag – Leffler function, and prove derivation of the properties. In Section 3, we obtain the inversion formula.

2. Preliminaries on Fractional Derivatives

2.1 Fractional derivative via fractional difference

Definition 1 : Let there be a continuous function $f : R \to R,\ t \to f(t)$ (but not necessarily differentiable). Let $h > 0$ be a constant discretization span. The forward operator $FW(h)$ is given by

$$FW(h) f(t) = f(t + h). \tag{13}$$

With regard to (13), the fractional difference of order $\alpha, 0 < \alpha < 1$, of the function $f(t)$ is given by

$$\Delta^{\alpha} f(t) = (FW - 1)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[t + (\alpha - k)h],$$

and the fractional derivative of order α is defined by the limit

$$f^{(\alpha)}(t) = \lim_{h \to 0} \frac{\Delta^{\alpha} f(t)}{h^{\alpha}} \quad . \tag{14}$$

2.2 Modified Riemann – Liouville fractional derivative

To overcome with some drawback with the Riemann – Liouville fractional derivative, the modified version is devised [10, 11].

Definition 2 : Let $f : R \to R, t \to f(t)$ is a continuous function.

(i) When $f(t)$ is constant K, its fractional derivative of order α, is given by

$$\begin{aligned} D_t^{\alpha} K &= K \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{t^{\alpha}}, \alpha \geq 0 \\ &= 0 \quad , \qquad \alpha > 0 \end{aligned}$$

(ii) For $f(t)$ being not a constant, we have

$$f(t) = f(0) + (f(t) - f(0))$$

and its fractional derivative is defined by

$$f^{(\alpha)}(t) = D_t^{\alpha} f(0) + D_t^{\alpha} (f(t) - f(0)) \quad , \tag{15}$$

which when $\alpha < 0$, is given by

$$D_t^{\alpha}(f(t) - f(0)) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t - \xi)^{-\alpha - 1} f(\xi) d\xi, \qquad \alpha < 0 \tag{16}$$

whereas for $\alpha > 0$, we have

$$D_t^{\alpha}(f(t) - f(0)) = D_t^{\alpha}(f(t)) = D_t(f^{(\alpha - 1)}(t)) \tag{17}$$

and

$$f^{(\alpha)}(t) = (f^{(\alpha - n)}(t))^{(n)} \quad , n \leq \alpha < n + 1 \, . \tag{18}$$

2.3 Taylor series of fractional order

Definition 3 : The continuous function $f : R \to R, t \to f(t)$ has a fractional derivative of order $k\alpha$. For any positive integer k and for any α, $0 < \alpha \leq 1$, we have

$$f(t+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(t) \quad , 0 < \alpha \leq 1, \tag{19}$$

where $\Gamma(1+\alpha k) = (\alpha k)!$.

2.4 Integration with respect to $(dt)^\alpha$

The integral with respect to is defined as the solution of the fractional differential equation

$$dy = f(t)(dt)^\alpha \quad , \quad t \geq 0, y(0) = 0 \tag{20}$$

Lemma 1 [10, 11] : Let $f(t)$be a continuous function. Then the solution $y(t), y(0) = 0$, is given by

$$\begin{aligned} y &= \int_0^t f(\xi)(d\xi)^\alpha \\ &= \alpha \int_0^t (t-\xi)^{(\alpha-1)} f(\xi) d\xi, \quad 0 < \alpha < 1 \ . \end{aligned} \tag{21}$$

3. Elzaki Transform of Fractional Order and the Mittag – Leffler Function

In this section Elzaki transform of fractional order is defined by using the Mittag – Leffler function, which is the generalization of the exponential function. Properties and convolution theorem are proved using the Elzaki transform of fractional order.

By virtue of terminologies used in the preceding sections and recalling those described for Fourier and the Lapalce transforms, respectively , through the Mittag – Leffler function [10, 11], following definition results.

Definition 4 : Let $f(t)$ be a function that vanishes for negative values of t . Then the Elzaki transform of order α, for finite $f(t)$, is defined by

$$\begin{aligned} \mathbb{E}_\alpha[f(t)] &= T_\alpha(v) = v\int_0^\infty E_\alpha\left(-\frac{t}{v}\right)^\alpha f(t)(dt)^\alpha & (22) \\ &= \int_0^\infty v^2 f(vt) E_\alpha(-t)^\alpha (dt)^\alpha & (23) \\ &= \lim_{M\uparrow\infty}\int_0^M v^2 f(vt) E_\alpha(-t)^\alpha (dt)^\alpha & (24) \end{aligned}$$

where E_α is the Mittag – Leffler function, given by (11).

Theorem 1 (Elzaki Laplace Duality of Fractional order) : If the Laplace transform of fractional order of a function $f(t)$ is $L_\alpha\{f(t)\} = F_\alpha(s)$ and the Elzaki transform $\mathbb{E}_\alpha[f(t)] = T_\alpha(v)$ is of order α, then

$$T_\alpha(v) = vF_\alpha\left(\frac{1}{v}\right) \tag{25}$$

Proof : Invoking the definition of Elzaki transform of fractional order (22), we write

$$\begin{aligned} \mathbb{E}_\alpha[f(t)] &= T_\alpha(v) = \int_0^\infty v\, f(t) E_\alpha\left(\left(-\frac{t}{v}\right)^\alpha\right)(dt)^\alpha \\ &= \lim_{M\uparrow\infty}\alpha\int_0^M (M-t)^{\alpha-1} v f(t) E_\alpha\left(\left(-\frac{t}{v}\right)^\alpha\right) dt\ . & (26) \end{aligned}$$

By using the change of variable $wv \to t$, we get the right hand side

$$= \lim_{M\uparrow\infty}\alpha\int_0^M (M-vw)^{\alpha-1} v^2 f(vw) E_\alpha(-w)^\alpha\, dw\ .$$

By using the change of variable $vw \to t'$, futher we get

$$\begin{aligned} &= \lim_{M\uparrow\infty}\alpha\int_0^M (M-t')^{\alpha-1} v^2 f(t') E_\alpha\left(-\frac{t'}{v}\right)^\alpha \frac{dt'}{v} \\ &= v\int_0^\infty f(t') E_\alpha\left(-\frac{t'}{v}\right)^\alpha dt' \quad \text{, using Laplace transform} \end{aligned}$$

Hence,

$$\mathbb{E}_\alpha[f(t)] = T_\alpha(v) = vF_\alpha\left(\frac{1}{v}\right) \tag{27}$$

proves the theorem.

Theorem 2 (Change of Scale Property) : Let $f(at)$ be a function in the set A , where a is non – zero constant. Then

$$\mathbb{E}_\alpha[f(at)] = \frac{1}{a^\alpha} T_\alpha\left(\frac{v}{a}\right) \quad . \tag{28}$$

Conditions are as mentioned above.

Proof : Using (22), we have

$$\begin{aligned} \mathbb{E}_\alpha[f(at)] &= \int_0^\infty v\ f(at) E_\alpha\left(\left(-\frac{t}{v}\right)^\alpha\right)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} v\ f(at) E_\alpha\left(-\frac{t}{v}\right)^\alpha dt \end{aligned} \tag{29}$$

By using the change of variable $at \to t'$, we get

$$= \lim_{M\uparrow\infty} \alpha \int_0^{Ma} (M - \frac{t'}{a})^{\alpha-1} v\ f(t') E_\alpha\left(-\frac{t'}{av}\right)^\alpha \frac{dt'}{a}$$

$$= \int_0^{Ma} \frac{(Ma-t')^{\alpha-1}}{a^\alpha} v\ f(t') E_\alpha\left(-\frac{t'}{av}\right)^\alpha dt'$$

i.e.

$$\mathbb{E}_\alpha[f(at)] = \frac{1}{a^\alpha} T_\alpha\left(\frac{v}{a}\right) \quad . \tag{30}$$

Theorem is proved.

Theorem 3 : Let $f(t-b)$ is a function of fractional Elzaki transform. Then

$$\mathbb{E}_\alpha[f(t-b)] = E_\alpha\left(\left(-\frac{b}{v}\right)^\alpha\right) T_\alpha(v) \quad . \tag{31}$$

Proof : By (22) of Definition 4, we have.

$$\begin{aligned} \mathbb{E}_\alpha[f(t-b)] &= \int_0^\infty v\ f(t-b) E_\alpha\left(\left(-\frac{t}{v}\right)^\alpha\right)(dt)^\alpha \\ &= \lim_{M\uparrow\infty} \alpha \int_0^{M} (M-t)^{\alpha-1} v\ f(t-b) E_\alpha\left(-\frac{t}{v}\right)^\alpha dt \quad . \end{aligned} \tag{32}$$

Considering $t - b = x$, we have the right hand side

$$= \lim_{M\uparrow\infty} \alpha \int_0^{M-b} (M-b-x)^{\alpha-1} v\ f(x) E_\alpha\left(-\frac{(b+x)}{v}\right)^\alpha dx$$

$$= \int_0^{M-b} (M-b-x)^{\alpha-1} v\ f(x) E_\alpha \left(-\frac{x}{v}\right)^\alpha E_\alpha \left(-\frac{b}{v}\right)^\alpha dx$$

i.e.

$$\mathbb{E}_\alpha[f(t-b)] = E_\alpha \left(-\frac{b}{v}\right)^\alpha T_\alpha(v) \quad , \tag{33}$$

which is due to $E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) E_\alpha(\lambda y^\alpha)$.

Theorem 4 : If $f(t)$ is $E_\alpha(a^\alpha t^\alpha) f(t)$, then the Elzaki transform is given by

$$\mathbb{E}_\alpha[E_\alpha(a^\alpha t^\alpha) f(t)] = \left(\frac{1}{1-av}\right)^\alpha T_\alpha \left(\frac{v}{1-av}\right) \quad . \tag{34}$$

Proof : Using (22) again of Definition 4, we have

$$\begin{aligned} \mathbb{E}_\alpha[E_\alpha(a^\alpha t^\alpha) f(t)] &= \int_0^\infty v\ E_\alpha(a^\alpha t^\alpha) f(t) E_\alpha \left(\left(-\frac{t}{v}\right)^\alpha\right) (dt)^\alpha \\ &= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} v\ f(t) E_\alpha(a^\alpha t^\alpha) E_\alpha \left(-\frac{t}{v}\right)^\alpha dt \end{aligned}$$

i.e.

$$= \lim_{M\uparrow\infty} \alpha \int_0^M (M-t)^{\alpha-1} v\ f(t) E_\alpha \left(-\left(\frac{t-avt}{v}\right)\right)^\alpha dt$$

Setting $(1-av)t = w$, we have the right hand side, reduced to

$$= \lim_{M\uparrow\infty} \alpha \int_0^{M-av} \left(M - \frac{w}{1-av}\right)^{\alpha-1} v\ f\left(\frac{w}{1-av}\right) E_\alpha \left(-\frac{w}{v}\right)^\alpha \frac{dw}{(1-av)}$$

$$= \int_0^{M-av} \left(\frac{1}{1-av}\right)^\alpha (M(1-av)-w))^{\alpha-1} v\ f\left(\frac{w}{1-av}\right) E_\alpha \left(-\frac{w}{v}\right)^\alpha dw \quad ,$$

i.e.

$$\mathbb{E}_\alpha[E_\alpha(a^\alpha t^\alpha) f(t)] = \left(\frac{1}{1-av}\right)^\alpha T_\alpha \left(\frac{v}{1-av}\right) \quad . \tag{35}$$

Hence, the theorem is proved.

Theorem 5 : Let the convolution of two functions $f(t)$ and $g(t)$ of order α is given by

$$(f(t) * g(t))_\alpha = \int_0^\infty f(t-\xi)g(\xi)(d\xi)^\alpha. \tag{36}$$

Then the convolution of Elzaki transform of order α is

$$\mathbb{E}_\alpha[(f(t) * g(t))_\alpha] = \frac{1}{v} M_\alpha(v) N_\alpha(v) \quad . \tag{37}$$

Proof : The convolution of Laplace transform of order α is given by

$$L_\alpha[(f(t) * g(t))_\alpha] = L_\alpha\{f(t)\} L_\alpha\{g(t)\} \tag{38}$$

Now using Elzaki – Laplace duality (Theorem 1, (25)), we have

$$\begin{aligned} \mathbb{E}_\alpha[(f(t) * g(t))_\alpha] &= vL_\alpha\{f(t)\}L_\alpha\{g(t)\} \\ &= v\left[F_\alpha\left(\frac{1}{v}\right)G_\alpha\left(\frac{1}{v}\right)\right] \quad , \text{ as } M_\alpha(v) = vF_\alpha\left(\frac{1}{v}\right) \\ &= v\left[\frac{M_\alpha(v)}{v}\frac{N_\alpha(v)}{v}\right] \quad . \end{aligned}$$

i.e.

$$\mathbb{E}_\alpha[(f(t) * g(t))_\alpha] = \frac{1}{v} M_\alpha(v) N_\alpha(v) \quad . \tag{39}$$

The theorem is proved.

Theorem 6 : The inversion formula of the Elzaki transform of fractional order α that is given by (22), is

$$f_\alpha(t) = \sum \text{Res } [\frac{1}{v} e^{t/v} \mathbb{E}_\alpha(v)] \quad . \tag{40}$$

According to the fractional Elzaki – Laplace duality (Theorem 1), the inversion formula can easily be proved.

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