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(ε, δ)-FREUDENTHAL KANTOR TRIPLE SYSTEMS, δ-STRUCTURABLE ALGEBRAS AND LIE SUPERALGEBRAS

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Abstract

In this paper we discuss (ε, δ)-Freudenthal Kantor triple systems with certain structure on the subspace L_{-2} of the corresponding standard embedding five graded Lie (super)algebra $L(\varepsilon, \delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $[L_i, L_j] \subseteq L_{i+j}$. We recall Lie and Jordan structures associated with (ε, δ)-Freudenthal Kantor triple systems ([26], [27]) and the give results for unitary and pseudo-unitary (ε, δ)-Freudenthal Kantor triple systems. Further, we give the notion of δ-structurable algebras and connect them to $(-1, \delta)$-Freudenthal Kantor triple systems and the corresponding Lie (super)algebra construction.

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1 Introduction

Nonassociative algebras are rich in algebraic structures and provide an important common ground for various branches of mathematics. Specially, Jordan and Lie (super)algebras plays an important role in many mathematical and physical subjects ([9]-[12], [14], [22], [24], [39], [43]). We note that the construction and characterization of these algebras can be expressed in terms of the notion of triple systems ([18], [40]) by using the standard embedding method ([20], [33], [34], [41], [45]). In particular, the generalized Jordan triple system of second order (that is, the $(-1, 1)$-Freudenthal Kantor triple system) is a useful concept ([12]-[19], [29]-[32], [44]) for the constructions of simple Lie algebras, while the $(-1, -1)$-Freudenthal Kantor triple system plays the same role ([5], [20], [21], [23]) for the construction of Lie superalgebras. The purpose of this paper is to recall Lie and Jordan structures associated with (ε, δ)-Freudenthal Kantor triple systems ([26], [27]) and the give results for unitary and pseudo-unitary (ε, δ)-Freudenthal Kantor triple systems. Further, we introduce the notion of δ-structurable algebras and connect them to $(-1, \delta)$-Freudenthal Kantor triple systems and the corresponding Lie (super)algebra construction.

2 Definitions, preliminaries, Lie and Jordan structures

2.1 (ϵ, δ)-Freudenthal Kantor triple systems, δ-Lie triple systems and Lie (super)algebras

We are concerned in this paper with triple systems which have finite dimension over a field Φ of characteristic $\neq 2$ or 3.

We recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \qquad (2)$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (1) and the identity

$$(abc) = (cba). \qquad (3)$$

We can generalize the concept of GJTS of 2nd order as follows (see [12], [13], [16], [20], [45] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \qquad (4)$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \qquad (5)$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \qquad (6)$$

is called an (ε,δ)-*Freudenthal Kantor triple system* (for short (ε,δ)-FKTS).

Remark. We note that $K(b,a) = -\delta K(a,b)$.

From now on we will mainly deal with this type of triple system.

An (ε,δ)-FKTS is said to be *balanced* if it fulfills $dim_\Phi\{K(a,b)\}_{span} = 1$.

An (ε,δ)-FKTS U is called *unitary* if the identity map Id is contained in $\kappa := K(U,U)$ i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i,b_i) = Id$.

Remark. A balanced triple system is unitary, since $Id \in \kappa = K(U,U)$.

Triple products are generally denoted (xyz), $\{xyz\}$, $[xyz]$ and $< xyz >$.

Remark. The concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method ([5], [12]-[16], [20], [21], [23], [31], [45]).

Remark. For an (ε,δ)-FKTS U we denote

$$S(a,b) := L(a,b) + \varepsilon L(b,a), \quad A(a,b) := L(a,b) - \varepsilon L(b,a), \qquad (7)$$

where $L(a,b)$ is defined by (6).

Remark. We note that $S(a,b) = \varepsilon S(b,a)$. Then $S(a,b)$ (respectively $A(a,b)$) is a derivation (respectively anti-derivation) of U ([26]), that is

$$[S(a,b), L(c,d)] = L(S(a,b)c,d) + L(c,S(a,b)d),$$
$$[A(a,b), L(c,d)] = L(A(a,b)c,d) - L(c,A(a,b)d).$$

For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc], a,b,c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \tag{8}$$

$a,b,x,y,z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS* ([13]).

Proposition 2.1 *([13],[20]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{9}$$

Corollary 2.1 *([13],[20]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product defined by*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}.$$

Thus we obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated to $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$, i.e.

$$D(T(\varepsilon,\delta),T(\varepsilon,\delta)) := \left\{\begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix}\right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{\begin{pmatrix} x \\ y \end{pmatrix}\middle| x,y \in U(\varepsilon,\delta)\right\}_{span}.$$

Remark. We note that $L(\varepsilon,\delta) := L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the five graded Lie (super)algebra, such that $L_{-1} \oplus L_1 = T(\varepsilon,\delta)$ and $D(T(\varepsilon,\delta),T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ with $[L_i, L_j] \subseteq L_{i+j}$.

Proposition 2.1 is generalized as follows.

Theorem 2.1 *([27]) Let $(U, (xyz))$ be a (ε, δ)-FKTS and P be an endomorphism of U such that $P^2 = \mu Id$ and $P(xyz) = \nu(Px)(Py)(Pz)$, $\mu, \nu = \pm 1$. Define the new triple product by*

$$[x, y, z] := x(Py)z + \varepsilon^* y(Px)z - \varepsilon^* x(Pz)y + \varepsilon^* \delta y(Pz)x,$$

where $\varepsilon^* = \mu\nu\varepsilon$. If $\mu\nu\varepsilon\delta = -1$, then $(U(\varepsilon, \delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$).

2.2 JTSs associated with (ε, δ)-FKTSs

In this section, we consider the properties of $\{K(x, y)\}_{span}$ defined by (6).

Proposition 2.2 *([26]) Let U be an (ε, δ)-FKTS. Then we have*

$$K(u, v)K(x, y) = L(v, K(x, y)u) - \delta L(u, K(x, y)v) \quad (10)$$
$$= \varepsilon\delta L(K(u, v)y, x) - \varepsilon L(K(u, v)x, y) \quad (11)$$

for any $u, v, x, y \in U$, where $L(x, y), K(x, y)$ are defined by (6).

Corollary 2.2 *([26]) The $K(x, y)$ satisfies a Lie relation type, that is*
$[K(x, y), [K(u, v), K(a, b)]] =$
$\delta K([K(u, v), K(a, b)]y, x) - K([K(u, v), K(a, b)]x, y)$
and the following identities are valid for $S(a, b)$ defined by (7):

$$[K(a, b), K(x, y)] = \delta S(x, K(a, b)y) - \varepsilon S(K(a, b)x, y), \quad (12)$$
$$[S(x, y), K(a, b)] = K(K(a, b)y, x) + \varepsilon K(K(a, b)x, y), \quad (13)$$
$$= K(S(x, y)a, b) + K(a, S(x, y)b), \quad (14)$$
$$[S(a, b), S(x, y)] = S(S(a, b)x, y) + S(x, S(a, b)y). \quad (15)$$

Remark. The Corollary 2.2 shows that the $K(a, b)$ defined by (6) has a structure of a LTS with respect to the product

$$[[K(a, b), K(c, d)], K(e, f)] = [K(a, b), K(c, d), K(e, f)].$$

Proposition 2.3 *([26]) Let U be an (ε,δ)-FKTS. Then the $K(x,y)$ satisfies a JTS relation type, that is*

$$K(a,b)K(x,y)K(u,v)+K(u,v)K(x,y)K(a,b)$$
$$=K(K(a,b)K(x,y)u,v)-\delta K(K(a,b)K(x,y)v,u), \tag{16}$$
$$=\varepsilon\delta K(K(a,b)x,K(u,v)y)-\varepsilon K(K(a,b)y,K(u,v)x). \tag{17}$$

Let $\kappa=\{K(x,y)|x,y\in U\}_{span}$ and define a triple product in κ by

$$\{K_1,K_2,K_3\}:=K_1K_2K_3+K_3K_2K_1,\ (K_j\in\kappa). \tag{18}$$

Remark. From Proposition 2.3 it follows then that

$$\{K(a,b),K(x,y),K(c,d)\}=K(K(a,b)K(x,y)c,d)-\delta K(K(a,b)K(x,y)d,c).$$

Proposition 2.4 *([26]) The triple product $\{K_1,K_2,K_3\}$ in (18) is a JTS.*

Proposition 2.5 *([26]) For the triple product $\{,\ ,\ ,\}$ defined by (18) let $\sigma(x,y)\in End\ \kappa$ and $\theta(x,y)\in End\ \kappa$, $x,y\in U$, be defined by*

$$\sigma(x,y)K(a,b):=K(K(a,b)x,y)-\varepsilon\delta K(x,K(a,b)y), \tag{19}$$

$$\theta(x,y)K(a,b):=K(K(a,b)x,y)+\varepsilon\delta K(x,K(a,b)y). \tag{20}$$

Then $\sigma(x,y)$ is a derivation and $\theta(x,y)$ is an anti-derivation of the JTS κ.

If we assume the unitary property, we have the following results:

Proposition 2.6 *Let U be an unitary (ε,δ)-FKTS. Then we have*

$$(i)\ \varepsilon=\delta\ (or\ \varepsilon\delta=1),$$

$$(ii)\ K(x,y)=L(y,x)-\varepsilon L(x,y)=-\varepsilon A(x,y), \tag{21}$$

$$(iii)\ K(x,y)K(u,v)+K(u,v)K(x,y)=K(K(u,v)x,y)+$$
$$K(x,K(u,v)y)=K(u,K(x,y)v)+K(K(x,y)u,v), \tag{22}$$

$$(iv)\ K(x,y)\text{ is an anti}-\text{derivation of }U.$$

Proposition 2.7 *([26]) Let U be an unitary (ε, δ)-FKTS. Then the commutative product in $K(U, U)$ defined by*

$$K(u,v) * K(x,y) = K(u,v)K(x,y) + K(x,y)K(u,v) = K(x, K(u,v)y) + K(K(u,v)x, y) \tag{23}$$

defines a Jordan algebra κ^*. Moreover, $\sigma(x,y)$ is a derivation of κ^*.

Remark. We note that κ's property is the same to the property of $L_{-2} = \{\begin{pmatrix} 0 & K(x,y) \\ 0 & 0 \end{pmatrix} | x, y \in U\}_{span}$, thus the investigation of the (ε, δ)-FKTS U means the study of the standard embedding Lie (super)algebra.

2.3 Pseudo unitarity

Let U be an (ε, δ)-FKTS. For $\varepsilon\delta = -1$, we can not have unitarity, so we note ([27]) that we can relax the condition as follows. Let $P \in \mathrm{End}U$ with

$$P^2 = \mu Id, \quad \mu = \pm 1, \tag{24}$$

$$P(xyz) = \nu(Px)(Py)(Pz), \quad \nu = \pm 1. \tag{25}$$

We note then that the triple system $(U, (xyz)^*)$ with the new triple product $(xyz)^* := x(Py)z$ is an (ε^*, δ)-FKTS with $\varepsilon^* = \mu\nu\varepsilon$ ([27]). We denote $(U, (xyz)^*)$ by U^*. Therefore, if $\varepsilon = -\delta$ and if we can choose $\mu\nu = -1$, then U^* is a (δ, δ)-FKTS, so that it can be unitary ([27]).

Let U be an (ε, δ)-FKTS and let $P \in \mathrm{End}U$. We call U to be *pseudo unitary* if there are ν, $\mu \in \{\pm 1\}$ and $P \in \kappa = K(U, U)$ (i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i, b_i) = P$) and P satisfies (24).

We show now that (25) is a consequence of $P \in \kappa$ and (24) by proving the following lemma end proposition.

Lemma 2.1 *Let U be an (ε, δ)-FKTS and let $P \in \mathrm{End}U$ such that $P \in \kappa = K(U, U)$. Then we have*

$$K(Px, y) = L(y, x)P - \varepsilon PL(x, y). \tag{26}$$

Proof. Indeed, we have $K(K(a, b)x, y) = L(y, x)K(a, b) - \varepsilon K(a, b)L(x, y)$, by (5). Put now $a = a_i, b = b_i$ in the last identity and summing over i we obtain (26).

Proposition 2.8 *Let U be an (ε, δ)-FKTS and let $P \in \mathrm{End}U$ such that $P \in \kappa = K(U,U)$ and (24) is fulfilled. Then we have*

$$P(xyz) = \mu\varepsilon\delta(Px)(Py)(Pz) = \mu\delta yx(Pz) + \mu yz(Px) - \mu\delta(Px)zy. \quad (27)$$

Furthermore, if $\mu\varepsilon\delta = 1$, then P is an automorphism of U.

Proof. By (26), we have $(Px)zy - \delta yz(Px) = yx(Pz) - \varepsilon P(xyz)$, or

$$P(xyz) = -\varepsilon(Px)zy + \varepsilon yx(Pz) + \varepsilon\delta yz(Px). \quad (28)$$

Applying P again to the identity (28) we have, by (24),

$$\mu xyz = P^2(xyz) = -\varepsilon P((Px)zy) + \varepsilon P(yx(Pz)) + \varepsilon\delta P(yz(Px)). \quad (29)$$

Changing $x \to Px$ and $y \leftrightarrow z$ in (28) and multiplying with $-\varepsilon$ follows $-\varepsilon P((Px)zy) = -\varepsilon[-\varepsilon(P^2x)yz + \varepsilon z(Px)(Py) + \varepsilon\delta zy(P^2x)]$ hence, by (24),

$$-\varepsilon P((Px)zy) = \mu xyz - z(Px)(Py) - \mu\delta zyx. \quad (30)$$

Changing $x \leftrightarrow y$ and $z \to Pz$ in (28) and multiplying with ε follows $\varepsilon P(yx(Pz)) = \varepsilon[-\varepsilon(Py)(Pz)x + \varepsilon xy(P^2z) + \varepsilon\delta x(Pz)(Py)]$ hence, by (24),

$$\varepsilon P(yx(Pz)) = -(Py)(Pz)x + \mu xyz + \delta x(Pz)(Py). \quad (31)$$

Finally, changing $x \to y \to z \to Px$ in (28) and multiplying with $\varepsilon\delta$ follows $\varepsilon\delta P(yz(Px)) = \varepsilon\delta[-\varepsilon(Py)(Px)z + \varepsilon zy(P^2x) + \varepsilon\delta z(Px)(Py)]$ hence, by (24),

$$\varepsilon\delta P(yz(Px)) = -\delta(Py)(Px)z + \mu\delta zyx + z(Px)(Py). \quad (32)$$

Substituting in the right hand side of (29) the identities (30), (31), (32) by straightforward calculations follows

$$\mu xyz = -\delta x(Pz)(Py) + \delta(Py)(Px)z + (Py)(Pz)x. \quad (33)$$

Substituting $x \to Px, y \to Py, z \to Pz$ in (33) follows $\mu(Px)(Py)(Pz) = -\delta(Px)(P^2z)(P^2y) + \delta(P^2y)(P^2x)(Pz) + (P^2y)(P^2z)(Px)$ hence, by (24), $(Px)(Py)(Pz) = -\mu\delta(Px)zy + \mu\delta yx(Pz) + \mu yz(Px)$, or equivalently

$$(Px)(Py)(Pz) = \mu\varepsilon\delta[-\varepsilon(Px)zy + \varepsilon yx(Pz) + \varepsilon\delta yz(Px)]. \quad (34)$$

Comparing (34) with (28) follows $(Px)(Py)(Pz) = \mu\varepsilon\delta P(xyz)$, or equivalently $P(xyz) = \mu\varepsilon\delta(Px)(Py)(Pz)$. The last line and (34) complete the proof of (27). Moreover, if $\mu\epsilon\delta = 1$ then clearly P is an automorphism of U.

Remark. The identity (25) is a direct consequence of (27) for $\nu = \mu\varepsilon\delta$.

Remark. If U is unitary, it can be also regarded as pseudo unitary by taking $\nu = \mu = 1$ and $P = Id$.

Proposition 2.9 *Let U be an (ε, δ)-FKTS. If U is pseudo unitary with ν, μ and P as before, then $\varepsilon\delta = \mu\nu$ and $K(x, y) = \mu L(y, Px)P - \mu\delta L(x, Py)P$.*

Proof. It follows directly from Proposition 2.8 and Proposition 3.1 of [27].

3 δ-structurable algebras

Our start point is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Hence within the general framework of (ϵ, δ)-FKTSs $(\epsilon, \delta = \pm 1)$ and the standard embedding Lie (super)algebra construction studied in [5],[6],[12]-[14], [23], we define δ-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [1], [2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to GJTSs 2nd order as introduced and studied in [29], [30] and further studied in [3], [4], [28], [35]-[38], [42]. Their importance lies with constructions of five graded Lie algebras. For $\delta = -1$ the anti-structurable algebras may similarly shed light on the notion of $(-1, -1)$-FKTSs hence (by [5], [6]) on the construction of Lie superalgebras.

Let $(\mathcal{A}, ^-)$ be a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x, \overline{xy} = \overline{y}\,\overline{x}, x, y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1. Since char$\Phi \neq 2$, by [1] we have $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathcal{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$.

Suppose $x, y, z \in \mathcal{A}$. Put $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. Note that

$$\overline{[x, y, z]} = -[\overline{z}, \overline{y}, \overline{x}]. \tag{35}$$

The operators L_x and R_x are defined by $L_x(y) := xy, R_x(y) := yx$.

For $\delta = \pm 1$ and $x, y \in \mathcal{A}$ define

$$^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta(R_x R_{\overline{y}} - R_y R_{\overline{x}}), \tag{36}$$

$$^{\delta}B_{\mathcal{A}}(x,y,z) :=^{\delta} V_{x,y}(z) = (x\overline{y})z + \delta[(z\overline{y})x - (z\overline{x})y], x, y, z \in \mathcal{A}. \tag{37}$$

$^{+}B_{\mathcal{A}}(x,y,z)$ is called the *triple system obtained from the algebra* $(\mathcal{A},^{-})$. We will call $^{-}B_{\mathcal{A}}(x,y,z)$ the *anti-triple system obtained from the algebra* $(\mathcal{A},^{-})$. We shall write for short $V_{x,y} :=^{\delta}V_{x,y}$, $B_{\mathcal{A}} := (^{\delta}B_{\mathcal{A}}, \mathcal{A})$.

A unital non-associative algebra with involution $(\mathcal{A},^{-})$ is called a *structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \tag{38}$$

for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u, v, x, y \in \mathcal{A}$, and we call $(\mathcal{A},^{-})$ an *anti-structurable algebra* if (38) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

If $(\mathcal{A},^{-})$ is structurable then, in the terminology of [30], the triple system $B_{\mathcal{A}}$ is called a *GJTS* and by [7], $B_{\mathcal{A}}$ is a GJTS of 2nd order.

If $(\mathcal{A},^{-})$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS*.

Put $T_x := V_{x,1}$ for $x \in \mathcal{A}$. Then, by (36), $T_x = L_x + \delta R_{x-\overline{x}}$, for $x \in \mathcal{A}$. In particular, $T_h = L_h$ for $h \in \mathcal{H}$.

Remarks. (i) If $u = h \in \mathcal{H}$ and $x, y \in \mathcal{A}$, (38) becomes

$$[L_h, V_{x,y}] = V_{hx,y} - V_{x,hy}. \tag{39}$$

(ii) Suppose $^{-}$ is the identity map and hence $\mathcal{A}$ is commutative. If $(\mathcal{A},^{-})$ is δ-structurable then $\mathcal{A}$ is a Jordan algebra, by [25]. Conversely, by [32]§3, any Jordan algebra satisfies (39) if $V_{x,y} = {}^{+}V_{x,y}$ for $x, y \in \mathcal{A}$, hence it is structurable. By ([25]), any Jordan algebra is anti-structurable if

$$((hx)y)z - h((xy)z) = (x(yh))z - (xy)(hz) \tag{40}$$

for $h, x, y, z \in \mathcal{A}$. Using commutativity then (40) e.g. can be written $[x, h, y]z = [xy, z, h]$. Clearly, (40) is fulfilled by an associative algebra.

For $s \in \mathcal{S}$ and $h \in \mathcal{H}$ we say that $(\mathcal{A},^{-})$ is $\mathcal{S}$ *skew-alternative* if $[s, x, y] = -[x, s, y]$ while $(\mathcal{A},^{-})$ is $\mathcal{H}$ *skew-alternative* if $[h, x, y] = -[x, h, y]$ for $x, y \in \mathcal{A}$. We shall remark that if $(\mathcal{A},^{-})$ is $\mathcal{S}$ skew-alternative then by [1]§1,

$$[s, x, y] = -[x, s, y] = [x, y, s], s \in \mathcal{S}, x, y \in \mathcal{A}, \tag{41}$$

while if $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative then by (35),

$$[h,x,y] = -[x,h,y] = [x,y,h], h \in \mathcal{H}, x,y \in \mathcal{A}. \tag{42}$$

Proposition 3.1 *([25]) If $(\mathcal{A},^-)$ is structurable, then $(\mathcal{A},^-)$ is $\mathcal{S}$ skew-alternative. If $(\mathcal{A},^-)$ is anti-structurable, then $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative.*

Remark. Let $(\mathcal{A},^-)$ be a δ-structurable algebra and let $\mathrm{Der}(\mathcal{A},^-)$ be the set of derivations of $\mathcal{A}$ that commute with $^-$. By [25], $T_{\mathcal{A}} \cap \mathrm{Der}(\mathcal{A},^-) = 0$ and so we may define the ***structure algebra*** $\mathrm{Str}(\mathcal{A},^-) := T_{\mathcal{A}} \oplus \mathrm{Der}(\mathcal{A},^-)$ which plays an important role in the study of structurable algebras ([1]) and may play a role in the structure study of anti-structurable algebras.

3.1 Examples

For examples of structurable algebras we refer to [1] and [2].
Remark. Let (B,U) and (B',U') be two triple systems. We say that a linear map μ of U into U' is a *homomorphism* if μ satisfies $\mu(B(x,y,z)) = B'(\mu(x),\mu(y),\mu(z)), x,y,z \in U$. Moreover, if μ is bijective, then μ is called an *isomorphism*. In this case (B,U) and (B',U') are said to be *isomorphic*.

Let $(A,^-)$ be a unital non-associative algebra over Φ with involution $^-$ and let $(A^{op},^-)$ denote the *opposite* algebra, i.e. the algebra with multiplication defined by $x \cdot_{op} y = yx, x,y \in A$, where in the right hand side of the equality the multiplication is done in A. The algebras $(A,^-)$ and $(A^{op},^-)$ are isomorphic under the map $x \mapsto \overline{x}$. Let us define

$$^{\delta}V^{op}_{x,y} := R_{R_x(\overline{y})} + \delta(L_x L_{\overline{y}} - L_y L_{\overline{x}}), \tag{43}$$

$$^{\delta}B^{op}_{\mathcal{A}}(x,y,z) :=^{\delta} V^{op}_{x,y}(z) = z(\overline{y}x) + \delta[x(\overline{y}z) - y(\overline{x}z)], x,y,z \in \mathcal{A}. \tag{44}$$

Then $\mathcal{A}$ is a δ-structurable algebra if and only if $\mathcal{A}^{op}$ is a δ-structurable algebra since clearly, $B^{op}_{\mathcal{A}}$ is the triple system obtained from the algebra $(\mathcal{A}^{op},^-)$, and so $B_{\mathcal{A}}$ and $B^{op}_{\mathcal{A}}$ are isomorphic under the map $x \mapsto \overline{x}$, by (37) and (44).
Examples. (i) $\mathcal{M}_{m,n}(\Phi)$ with the product

$$\{x,y,z\} := xy^{\top}z + \delta(zy^{\top}x - zx^{\top}y) \tag{45}$$

where $x, y, z \in \mathcal{M}_{m,n}(\Phi)$, is a $(-1, \delta)$-FKTS ([26]). Hence $\mathcal{M}_{n,n}(\Phi)$ with the involution $x \mapsto x^\top$ is a δ-structurable algebra.

(ii) $\mathcal{M}_{m,n}(\mathbf{C})$ with the product $\{x, y, z\} := x\overline{y}^\top z + \delta(z\overline{y}^\top x - z\overline{x}^\top y)$, where $x, y, z \in \mathcal{M}_{m,n}(\mathbf{C})$, is a $(-1, \delta)$-FKTS ([26]). Hence $\mathcal{M}_{n,n}(\mathbf{C})$ with the involution $x \mapsto \overline{x}^\top$ is a δ-structurable algebra.

Remark. By [23], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U(-1,-1) := \mathcal{M}_{2n,m}(\Phi)$ with the product (45) then the corresponding standard embedding Lie superalgebra is $osp(2n|2m) = D(n, m)$ ([11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n,2n}(\Phi)$ is $osp(2n|4n)$. Similarly, if $U(-1,-1) := \mathcal{M}_{2n+1,m}(\Phi)$ with the product (45) then the corresponding standard embedding Lie superalgebra is $osp(2n+1|2m) = B(n, m)$ ([11]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n+1,2n+1}(\Phi)$ is $osp(2n+1|4n+2)$.

3.2 Anti-structurable algebras and extended Dynkin diagrams

In this section we give a correspondence between anti-structurable algebras and extended Dynkin diagrams as follows.

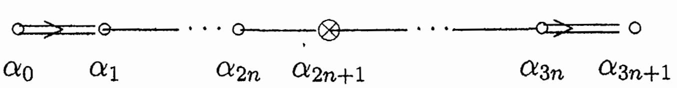
Let $U := \mathcal{M}_{l,l}(\Phi)$ with the product (45) and $\delta = -1$, that is

$$\{x, y, z\} := xy^\top z - zy^\top x + zx^\top y. \tag{46}$$

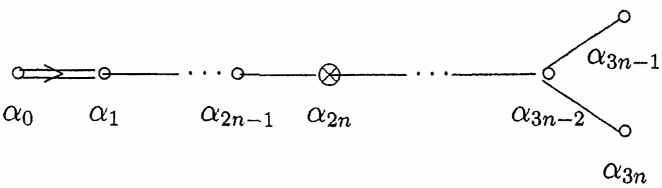
Then we obtain the standard embedding Lie superalgebra as follows; Lie (super)algebra notations and extended Dynkin diagrams are those of [8].

Proposition 3.2 *([26]) Let $(U, \{\ \ \})$, $U = \mathcal{M}_{l,l}(\Phi)$, be a simple unitary $(-1,-1)$-FKTSs defined by formula (46) and $L(U) = \oplus_{l=-2}^{2} L_l$ be the corresponding standard embedding Lie superalgebra. Then $L(U)$, $L_{-2} \oplus L_0 \oplus L_2$, L_0 and the corresponding extended Dynkin diagrams with $\otimes$ roots deleted are*

$$i) \begin{cases} L(U) = B(n, l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus B_n, & \text{for} \quad l = 2n+1, \\ L_0 = A_{l-1} \oplus B_n \oplus \Phi H \end{cases}$$



$$ii) \begin{cases} L(U) = D(n,l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus D_n, \quad \text{for} \quad l = 2n. \\ L_0 = A_{l-1} \oplus D_n \oplus \Phi H \end{cases}$$



Remark. These results mean that study of the correspondence between anti-structurable algebras and extended Dynkin diagrams is an useful concept for the structure theory of triple systems.

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