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METRIC TENSOR IS GROUPOID

Zbigniew Oziewicz
Universidad Nacional Autónoma de México
Facultad de Estudios Superiores Cuautitlán
oziewicz@unam.mx, oziewicz.zbigniew@gmail.com

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Abstract

Metric tensor is defined as an isomorphism from a module of vector fields (from ring derivations) to a module of differential forms. In fact metric tensor is Grassmann algebra isomorphism, because there are two Grassmann algebras, Grassmann algebra of contra-variant tensors and Grassmann algebra of covariant tensors. Thus metric tensor is a groupoid category. The Minkowski metric can be GL-lifted, to arbitrary material medium and to arbitrary metric, and this GL-lift unify the special and general relativities [Santilli 2010, 2015]. The Lorentz transformations, metric tensor automorphism group, hold well for arbitrary metric tensor, and thus for arbitrary Christoffel-Riemann curvature tensor.

Keywords: GL-lift of Minkowski metric tensor, Lorentz group for arbitrary metric tensor, groupoid category

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1 Metric tensor play a double role

- Metric tensor g is measuring distances in affine spacetime.
- Elwin Bruno Christoffel (1829-1900) in 1869 invented the derivation of the tensor algebra, known as the covariant derivation. The Christoffel derivation ∇ commute with the evaluation/contraction. Tensor g is a gravitational potential for the Christoffel derivation, $\nabla g = 0$.

Above interpretations of a metric tensor are *missing* the most important primordial definition of the metric tensor: it is wave - particle duality. Gregorio Ricci of Pisa, discovered before 1900, the most important duality in the Universe, that the Universe consists of two kinds of tensors, the contra-variant tensors, and covariant tensors. The contra-variant tensors are generated by ring-derivations and therefore should be associated with particle-face. The covariant tensors are owners of crests, fronts, and wave-longitudes, and therefore must be associated with wave-face of Universe. A metric tensor should be defined as an iso-morphisms from an algebra of contra-variant tensors to an algebra of covariant tensors. Therefore the genuine definition of the metric-tensor is ambient-dependent particle-wave iso-duality.

Metric tensor g is defined as ambient-dependent (refraction-dependent, gravity-dependent) **isomorphism** of tensor-algebras. It is particle-wave duality. And this most important duality of the Universe was discovered by Italian mathematician Gregorio Ricci-Curbastro (1853-1925) at Padua University before 1900. Posteriorly this duality in less clarity was postulated by Duc Louis De Broglie in his Doctoral Thesis in 1924, and another version of this duality was given by Arthur Compton in 1923.

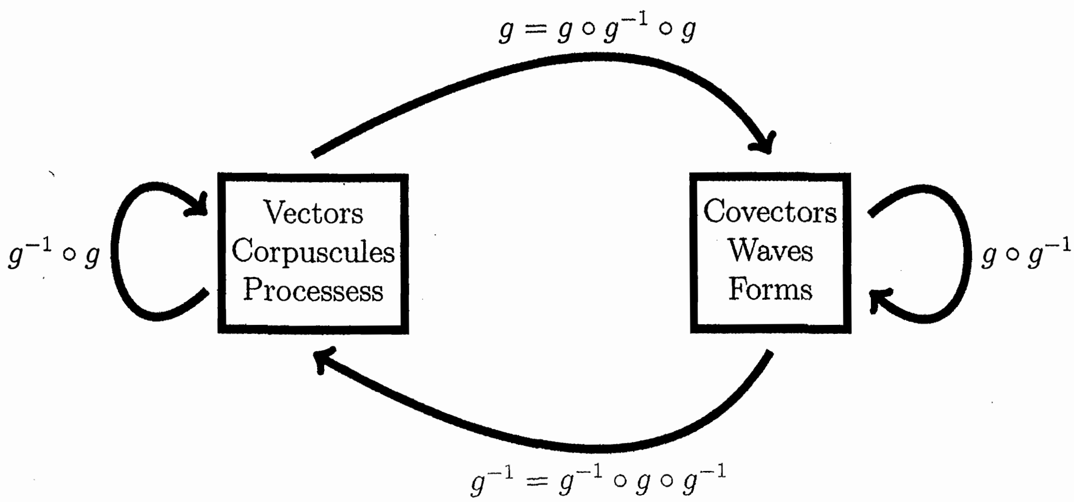
1.1 Definition (Metric tensor is a groupoid). Let $\mathcal{F}$ denotes an algebra of scalar fields. A metric tensor g is defined as an $\mathcal{F}$-module isomorphism from a $\mathcal{F}$-module of Leibniz derivations, $\operatorname{der}\mathcal{F}$, to $\mathcal{F}$-module of differential forms, giving rise to groupoid category, Figure 1. If $M \in \operatorname{der}\mathcal{F}$ is a Leibniz derivation, then gM denotes a g-companion medium-dependent differential form. A scalar field, $(gM)N$, is the evaluation of a differential form gM with a vector field N. Within groupoid as opposed to the group: $g^{-1} \circ g \quad \neq \quad g \circ g^{-1}$.

Split or decomposition of a metric tensor. Let $M \in \operatorname{der} \mathcal{F}$ be non-isotropic vector field, then

$$g = \frac{(gM) \otimes (gM)}{(gM)M} + g_M, \qquad g_M \in \operatorname{iso}\{\ker(gM), \ker M\}.$$
$$g^{-1} = \frac{M \otimes M}{(gM)M} + (g_M)^{-1}, \qquad (g_M)^{-1} \in \operatorname{iso}\{\ker M, \ker(gM)\}.$$

$$g \circ g^{-1} = \frac{(gM) \otimes M}{(gM)M} + g_M \circ (g_M)^{-1}$$
$$\neq \quad \frac{M \otimes (gM)}{(gM)M} + (g_M)^{-1} \circ g_M = g^{-1} \circ g.$$

Figure 1: A metric tensor is a groupoid categoty. This H. Brandt groupoid consists of four categorical arrows: $\{g, \quad g^{-1}, \quad g^{-1} \circ g, \quad g \circ g^{-1}\}$.



2 Santilli's conceptual unification of general relativity with special relativity

The meaning of the adjective 'relativity' within the *general relativity* is empty, because no concept is declared to be *relative*, *i.e.* the reference dependent, within such gravity theory. The lack of *relativity concepts* in the presence of gravity have been seen in the absence of analogy of the Lorentz transformations being a fundamental concept of special relativity in the absence of the gravity, and in the absence of a material medium.

One of the most important conceptual and philosophical problem was the lack of compatibility of non-vanishing curvature and presence of material medium, with the special relativity of vacuum and absence of gravity [Santilli 2010; 2015]. It is wide spread wrong belief (a strong preconception) that the necessary and sufficient condition on metric **tensor** that assure the existence of the Lorentz transformations (as the symmetry of this metric **tensor**) must be vanishing of the Christoffel curvature.

$$\text{morphism} \quad g \quad \xrightarrow{GL-\text{action/lift}} \quad U^{*-1} \circ g \circ U^{-1} \quad \begin{cases} \text{new lifted} \\ \text{morphism} \end{cases} \tag{2.1}$$

2.1 Definition (Deformation tensor)**.** Let $U \in GL$. In elasticity theory the GL-dependent tensor, $U^{*-1} \circ g \circ U^{-1}$, is said to be the transformed geometry, and the difference, $(U^{*-1} \circ g \circ U^{-1}) - g$, is said to be the deformation tensor.

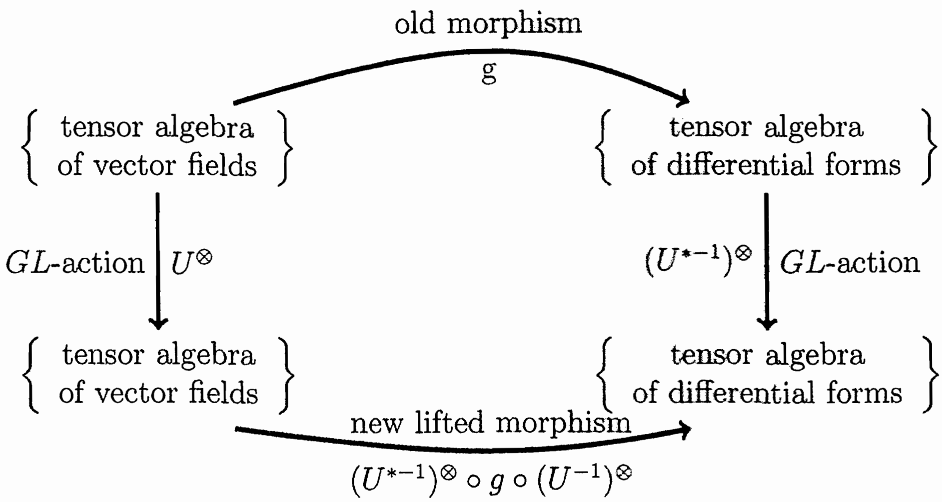
2.2 Clarification (Lifted geometry)**.** Santilli resolved the lack of compatibility of general relativity with special relativity by observing that the Minkowski metric η can be GL-lifted to arbitrary metric g within arbitrary dense medium and arbitrary curvature with arbitrary gravity field [Santilli 2010, 2015], Figure 2,

$$\eta \quad \xrightarrow{U \in GL\text{-lift}} \quad g = U^{*-1} \circ \eta \circ U^{-1} \tag{2.2}$$

Mendel Sachs re-named the GL-group as the Einstein group of general relativity [Sachs 1982].

The Santilli lift (2.2) gives the unification of special and general relativities. The Santilli GL-lift give rise to isotopy maps of all concomitant tensors depending on metric tensor. Isotopy is not GL-covariance because is acting only on metric tensor, leaving all vectors and other tensors intact. For

Figure 2: $GL(\mathcal{F})$-action (non-canonical or non-unitary lift) on **morphism** between tensors. A group action $U \in GL(\mathcal{F})$ and pull-back (transposition) U^* **cannot** be composed. This GL-action was proposed by Santilli as unifying special relativity of vacuum (of absence of gravity and absence of material dense medium) with general relativity of arbitrary material environment (medium) and of arbitrary gravity field.



example, for $X \in \operatorname{der}\mathcal{F}$,

$$(\eta X)X \xrightarrow{\text{isotopy}} (gX)X = \{(U^{-1*} \circ \eta \circ U^{-1})X\}X = (\eta U^{-1}X)(U^{-1}X) \quad (2.3)$$

For a sets of differential forms $\{\alpha^\mu\}$ and $\{\beta_\mu\}$, that do not need to be related to Cartan's concept of the moving frames because the metric tensor, as every tensor, is frame-free, co-frame-free, - one can present the metric **tensor** as the sum, with $GL(\mathcal{F})$-action for $U \in GL(\mathcal{F})$,

$$g = \sum \alpha^\mu \otimes \beta_\mu \xrightarrow{GL-\text{action};\quad GL-\text{lift}} (U^{*-1} \otimes U^{*-1})\, g \qquad (2.4)$$

The map from the tensor field g (2.4) - to $\mathcal{F}$-module **morphism** $\bar{g}$ from $\operatorname{der}\mathcal{F}$ to $(\operatorname{der}\mathcal{F})^*$, - is the composition of the annihilation operator represented by the evaluation, with creation operator, $\wedge$ or $\otimes$,

$$g = \alpha \otimes_{\mathcal{F}} \beta + \ldots \qquad \longleftrightarrow \qquad \bar{g} = \wedge_\alpha \circ \mathrm{ev}_\beta + \ldots \qquad (2.5)$$

The above morphism $\bar{g}$ extends to algebra map. The **morphism** $\bar{g}$ between tensors possesses compatible $GL(\mathcal{F})$-action,

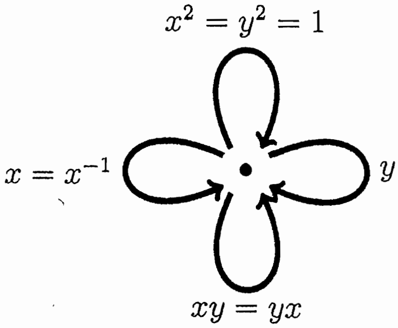
$$\bar{g} \xrightarrow{GL-\text{action}} U^{*-1} \circ \bar{g} \circ U^{-1} \tag{2.6}$$

One can re-derive the $GL(\mathcal{F})$-action (2.6) starting from the correspondence (2.5). The evaluation ev_β is a $\mathcal{F}$-module morphism from $\text{der}\,\mathcal{F}$ to $\mathcal{F}$. Whereas the creation $\wedge_\alpha$ is a $\mathcal{F}$-module morphism from $\mathcal{F}$ to $(\text{der}\,\mathcal{F})^*$. This leads to the following $GL(\mathcal{F})$-actions compatible with (2.6),

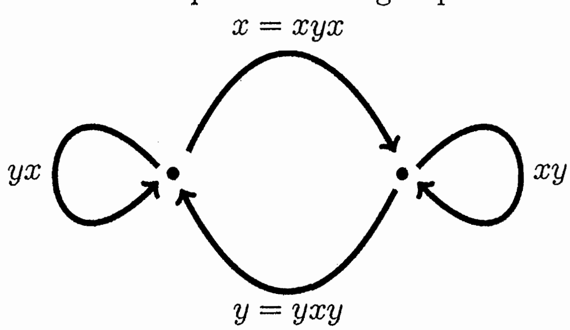
$$\begin{aligned}
\text{ev}_\beta &\xrightarrow{GL-\text{action}} \text{ev}_\beta \circ U^{-1} = \text{ev}_{U^{-1*}\beta} \\
\wedge_\alpha &\xrightarrow{GL-\text{action}} U^{*-1} \circ \wedge_\alpha = \wedge_{U^{-1*}\alpha} \circ U^{*-1} \\
\wedge_\alpha \circ \text{ev}_\beta &\xrightarrow{GL-\text{action}} U^{*-1} \circ \wedge_\alpha \circ \text{ev}_\beta \circ U^{-1}
\end{aligned} \tag{2.7}$$

2.3 Side remark (Pedagogical example). An illustrative example of four-element group (with unique unit) versus four-element groupoid (with **two** units).

Felix Klein Group $\mathbb{Z}_2 \times \mathbb{Z}_2$



Heinrich Brandt 1927
Groupoid is not a group



GROUP

	1	x	y	xy
x² ≡ 1	1	x	y	xy
x	x	1	xy	y
y	y	yx	1	x
xy	xy	y	x	1

GROUPOID

	xy	x	y	yx
xy	xy	-	y	-
x	x	-	yx	-
y	-	xy	-	y
yx	-	x	-	yx

2.4 Definition (Reference material body). A material body *at rest* is said to be the **reference** body. To be at rest is tautology to possess the **zero** velocity. The relative concepts depend on a choice of a body at rest, depend on a choice of a **reference** body.

Relative concepts, like a relative velocity and an electric field, were well known in a pre-relativity era. Hermann Minkowski introduced new **absolute** concepts, the absolute concept of a time-like body (as an iso-position process preserving positions), the absolute concept of an event as an element of affine spacetime, absolute electromagnetic field, absolute energy-momenta stress-tensor, etc. An absolute affine space-time of events was independently also advocated by Alfred North Whitehead (1861-1947).

A time-like body is reference-free, basis-free, frame-free and coordinate-free, and as well an event is reference-free and coordinate-free. For this reason the relativity theory should be renamed as the theory of absoluteness.

The philosophical essence of the colloquial relativity theory is a dichotomy:

$$\text{Dichotomy:} \qquad \textbf{absolute} \longleftrightarrow \textbf{relative}.$$

2.5 Example (Irrelevant dual frames). The basis-free and coordinate-free metric **tensor** is confusingly and misleadingly 'defined' in terms of irrelevant coordinates, coordinates that are mortal for a concept of a metric **tensor**. Let X be an algebra derivation,

$$g_{\mu\nu} \equiv (g\partial_\mu)\partial_\nu = g(\partial_\mu \otimes \partial_\nu) = \mathrm{ev}(g \otimes \partial_\mu \otimes \partial_\nu), \tag{2.8}$$

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad \Longrightarrow \quad gX = (g_{\mu\nu}\, dx^\mu \otimes dx^\nu)X = (g_{\mu\nu} X x^\nu) dx^\mu. \tag{2.9}$$

In (2.9) $\{dx^\mu\}$ are differential forms, irrelevant co-frames, that in some textbooks are interpreted as the **scalar** infinitesimal displacements that leads to 'old-fashioned' misleading interpretation of a metric **tensor** as an infinitesimal **scalar** distance squared denoted by ds^2. Such innocent notation is the source of mis-understandings, because does not exists neither a scalar 's' nor the differential ds, and $g \neq ds \otimes ds$. The Riemannian metric (2.9) is the genuine $GL(\mathcal{F})$-tensor.

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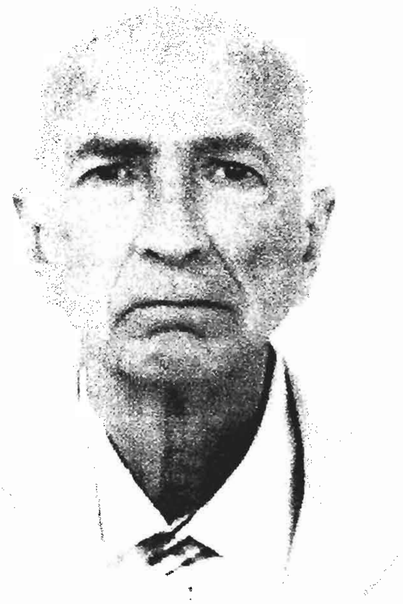

Zbigniew Oziewicz born in Wilno on August 22, 1941. Master in Theoretical Physics San Petersburg University, Rusia, 1964. Ph.D. in Mathematical Physics Wroclaw University Poland 1970. Habilitation in Mathematical Physics Wroclaw University Poland 1985.

Currently full professor at Universidad Nacional Autonoma de Mexico. Taught at the University of Wroclaw, at Texas University at San Antonio. at Gannon University in Erie Pennsylvania.

Research interest: category theory, relativity theory within groupoid category competing with Lorentz transformations.