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ADDITIVITY OF JORDAN n-TUPLE HIGHER DERIVABLE MAPS ON RINGS

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Abstract

Let $\mathscr{R}$ be a ring and $D = \{d_m\}_{m\in\mathbb{N}}$ a family of maps $d_m : \mathscr{R} \to \mathscr{R}$ satisfying $d_0 = I_{\mathscr{R}}$ (the identity map on $\mathscr{R}$) and $d_m(a_n \circ (\cdots (a_2 \circ a_1) \cdots)) = \sum_{p_1+\ldots+p_n=m} d_{p_n}(a_n) \circ (\cdots (d_{p_2}(a_2) \circ d_{p_1}(a_1)) \cdots)$, for all $a_1, \cdots, a_n \in \mathscr{R}$ and for each $m \in \mathbb{N}$, where $a \circ b = ab + ba$ is the Jordan product of a and b in $\mathscr{R}$. We prove that if $\mathscr{R}$ contain a non-trivial idempotent satisfying some conditions, then d_m is additive for each $m \in \mathbb{N}$. In particular, if $\mathscr{R}$ is a standard operator algebra, then d_m is additive for each $m \in \mathbb{N}$.

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1 Introduction

A ring $\mathscr{R}$ is called *k-torsion free* if $kx = 0$ implies $x = 0$, for any $x \in \mathscr{R}$, where k is a positive integer. We define the *Jordan product* $a \circ b$ of elements a, b in $\mathscr{R}$ as $a \circ b = ab + ba$.

Let $\mathscr{R}$ be a ring and $\delta : \mathscr{R} \to \mathscr{R}$ be a map. We call δ a *derivable map* or *multiplicative derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathscr{R}$, a *Jordan derivable map* or *multiplicative Jordan derivation* if $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$ for all $a, b \in \mathscr{R}$ and a *Jordan n-tuple derivable map* or *multiplicative Jordan n-tuple derivation* if

$$\delta(a_n \circ (\cdots (a_2 \circ a_1) \cdots)) = \sum_{k=1}^{n} a_n \circ (\cdots (\delta(a_k) \circ (\cdots (a_2 \circ a_1) \cdots)) \cdots)$$

for all $a_1, \cdots, a_n \in \mathscr{R}$.

Let $\mathscr{R}$ be a ring and $D = \{d_m\}_{m \in \mathbb{N}}$ a family of maps $d_m : \mathscr{R} \to \mathscr{R}$, for all $m \in \mathbb{N} = \{0, 1, 2, \cdots\}$, such that $d_0 = I_{\mathscr{R}}$ (the identity map on $\mathscr{R}$). We call D a *Jordan n-tuple higher derivable map* if

$$\begin{aligned} d_m(a_n \circ (\cdots (a_2 \circ a_1) \cdots)) \\ = \sum_{p_1 + \ldots + p_n = m} d_{p_n}(a_n) \circ (\cdots (d_{p_2}(a_2) \circ d_{p_1}(a_1)) \cdots), \quad (1) \end{aligned}$$

for all $a_1, \cdots, a_n \in \mathscr{R}$ and for each $m \in \mathbb{N}$.

We say that a Jordan n-tuple higher derivable map $D = \{d_m\}_{m \in \mathbb{N}}$ is *additive* if d_m is additive, for each $m \in \mathbb{N}$.

The authors in [3] studied the additivity of Jordan 2-tuple derivable maps, defined on rings having at least one non-trivial idempotent, and the authors in [2] extended their results to the Jordan n-tuple derivable maps, for all integers $n \geq 3$ and similar classes of rings. Recently, the authors in [1] studied the additivity of Jordan 2-tuple higher derivable maps for the same class of rings in [3]. They proved the following main theorem:

Theorem 1.1. *[1] Let $\mathscr{R}$ be a ring containing a nontrivial idempotent and satisfying the following conditions for $i, j, k \in \{1, 2\}$.*

(P1) If $a_{ij}x_{jk} = 0$ for all $x_{jk} \in \mathscr{R}_{jk}$, then $a_{ij} = 0$;

(P2) If $x_{ij}a_{jk} = 0$ for all $x_{ij} \in \mathscr{R}_{ij}$, then $a_{jk} = 0$;

(P3) If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ for all $x_{ii} \in \mathscr{R}_{ii}$, then $a_{ii} = 0$.

If the family $D = \{d_m\}_{m\in\mathbb{N}}$ of mappings $d_m : \mathscr{R} \to \mathscr{R}$ such that $d_0 = I_{\mathscr{R}}$ satisfies

$$d_m(ab + ba) = \sum_{p+q=m} d_p(a)d_q(b) + d_p(b)d_q(a)$$

for all $a, b \in \mathscr{R}$ and for each $m \in \mathbb{N}$, then $D = \{d_m\}_{m\in\mathbb{N}}$ is additive. In addition, if $\mathscr{R}$ is 2-torsion free, then $D = \{d_m\}_{m\in\mathbb{N}}$ is a Jordan higher derivation.

The aim of this paper is to extend the Theorem 1.1 to the class of Jordan n-tuple higher derivable maps, for all integer $n \geq 3$ and similar classes of rings in [2].

2 The main result

Our main result reads as follows.

Theorem 2.1. *Let $\mathscr{R}$ be a ring 2 and $(2^{n-1} - 1)$-torsion free containing a non-trivial idempotent e_1 and satisfying the following conditions:*

(i) $e_iae_j\mathscr{R}e_k = 0$ or $e_k\mathscr{R}e_iae_j = 0$ implies $e_iae_j = 0$ $(1 \leq i, j, k \leq 2)$;

(ii) $r_{22} \circ (\cdots(a_{22} \circ r_{22})\cdots) = 0$ for all $r_{22} \in \mathscr{R}_{22}$, implies $a_{22} = 0$.

Then every Jordan n-tuple higher derivable map $D = \{d_m\}_{m\in\mathbb{N}}$ is additive.

Following the ideas and techniques used by Ashraf et al. [1] and Ferreira et al. [2] we shall organize the proof of Theorem 2.1 in a series of lemmas.

We note that for $m = 1$, the identity (1) reduces to

$$d_1(a_n \circ (\cdots(a_2 \circ a_1)\cdots)) = \sum_{i=1}^{n} a_n \circ (\cdots(d_1(a_i) \circ (\cdots(a_2 \circ a_1)\cdots))\cdots)$$

for all $a_1, \cdots, a_n \in \mathscr{R}$. Thus, from the [2, Theorem 2.1.] d_1 is an additive map. We use this result throughout the paper without further reference.

We begin with the following lemma:

Lemma 2.1. *$d_m(0) = 0$ for each $m \in \mathbb{N}$.*

Proof. We prove the lemma based on second principle of mathematical induction for m. Let's assume that $d_k(0) = 0$ for every non-negative integer k with $k < m$ for any non-negative integer m. It follows that

$$\begin{aligned} d_m(0) &= d_m(\underbrace{0 \circ (\cdots (0 \circ 0) \cdots)}_{n-\text{times}}) \\ &= \sum_{p_1 + \ldots + p_n = m} d_{p_n}(0) \circ (\cdots (d_{p_2}(0) \circ d_{p_1}(0)) \cdots) \\ &= 0 \circ (\cdots (d_m(0) \circ 0) \cdots) \\ &\quad + \sum_{\substack{p_1 + \ldots + p_n = m \\ p_2 \neq m}} d_{p_n}(0) \circ (\cdots (d_{p_2}(0) \circ d_{p_1}(0)) \cdots) = 0. \end{aligned}$$

Therefore, we can conclude that $d_m(0) = 0$, for each nonnegative integer m. □

In what follows, we assume the assumption that d_k is an additive map for each non-negative integer $k < m$ for any non-negative integer m.

The following lemma will be used throughout this paper, whose proof is elementary and therefore omitted.

Lemma 2.2. *For any elements $a_{11} \in \mathscr{R}_{11}$, $b_{12} \in \mathscr{R}_{12}$, $c_{21} \in \mathscr{R}_{21}$ and $d_{22} \in \mathscr{R}_{22}$ the following holds*

$$\begin{aligned} & d_m(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots ((a_{11} + b_{12} + c_{21} + d_{22}) \circ r^1_{i_1 j_1}) \cdots)) \\ = & \sum_{p_1 + \ldots + p_n = m} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}}) \circ (\cdots (d_{p_k}(r^k_{i_k j_k}) \\ & \qquad \circ (\cdots (d_{p_2}(a_{11} + b_{12} + c_{21} + d_{22}) \circ d_{p_1}(r^1_{i_1 j_1})) \cdots)) \cdots) \\ = & \; r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots (d_m(a_{11} + b_{12} + c_{21} + d_{22}) \circ r^1_{i_1 j_1}) \cdots) \\ & + \sum_{\substack{p_1 + \ldots + p_n = m \\ p_2 \neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}}) \circ (\cdots (d_{p_k}(r^k_{i_k j_k}) \\ & \qquad \circ (\cdots (d_{p_2}(a_{11}) \circ d_{p_1}(r^1_{i_1 j_1})) \cdots)) \cdots) \end{aligned}$$

$$+\sum_{\substack{p_1+\ldots+p_n=m\\p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)$$
$$+\sum_{\substack{p_1+\ldots+p_n=m\\p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(c_{21})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)$$
$$+\sum_{\substack{p_1+\ldots+p_n=m\\p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(d_{22})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots) \qquad (2)$$

for all $r^k_{i_kj_k}\in\mathscr{R}_{i_kj_k}$ $(i_k,j_k=1,2;k=1,\cdots n-1)$.

Lemma 2.3. *For any elements* $a_{11}\in\mathscr{R}_{11}$, $b_{12}\in\mathscr{R}_{12}$, $c_{21}\in\mathscr{R}_{21}$ *and* $d_{22}\in\mathscr{R}_{22}$ *the following hold: (i)* $d_m(a_{11}+b_{12})=d_m(a_{11})+d_m(b_{12})$; *(ii)* $d_m(a_{11}+c_{21})=d_m(a_{11})+d_m(c_{21})$; *(iii)* $d_m(b_{12}+d_{22})=d_m(b_{12})+d_m(d_{22})$ *and (iv)* $d_m(c_{21}+d_{22})=d_m(c_{21})+d_m(d_{22})$.

Proof. For any elements $r^k_{i_kj_k}\in\mathscr{R}_{i_kj_k}$ $(k=1,\cdots,n-1)$, with $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, or $(i_k,j_k)=(2,2)$ $(k=1,\cdots,n-1)$, or $(i_1,j_1)=(1,2)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, we have

$$\begin{aligned}
& d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots((a_{11}+b_{12})\circ r^1_{i_1j_1})\cdots))\\
= {} & d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(a_{11}\circ r^1_{i_1j_1})\cdots))\\
& +d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(b_{12}\circ r^1_{i_1j_1})\cdots))\\
= {} & \sum_{p_1+\ldots+p_n=m} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})\\
& \qquad\circ(\cdots(d_{p_2}(a_{11})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)\\
& +\sum_{p_1+\ldots+p_n=m} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})\\
& \qquad\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)\\
= {} & r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(d_m(a_{11})\circ r^1_{i_1j_1})\cdots)
\end{aligned}$$

$$+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(a_{11})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)$$
$$+r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(d_m(b_{12})\circ r^1_{i_1j_1})\cdots)$$
$$+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots). \tag{3}$$

Considering $c_{21} = d_{22} = 0$ in (2) and subtracting (2) from (3), we obtain

$$r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots((d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))\circ r^1_{i_1j_1})\cdots)=0. \tag{4}$$

This implies that, if $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, then we obtain

$$r^1_{11}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))r^2_{22}\cdots r^{n-1}_{22}$$
$$+r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))r^1_{11}=0,$$

by (4), which yields

$$r^1_{11}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))r^2_{22}\cdots r^{n-1}_{22}$$
$$=r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))r^1_{11}=0.$$

It follows that $(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))_{12} = (d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))_{21} = 0$, by [2, Lemma 2.1.(i)]. If $(i_k, j_k) = (2,2)$ for all $(k = 1, \cdots, n-1)$, then by (4), we obtain

$$r^{n-1}_{22}\circ(\cdots((d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))\circ r^1_{22})\cdots)=0$$

which yields $(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))_{22} = 0$, by [2, Lemma 2.1.(ii)]. If $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-2)$, then by (4) again and the previous cases, we obtain

$$(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))r^1_{12}r^2_{22}\cdots r^{n-1}_{22}=0$$

which shows that $(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}))_{11} = 0$, by [2, Lemma 2.1.(i)]. Consequently, $d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12}) = 0$.

Similarly, we prove (ii), (iii) and (iv). □

Lemma 2.4. *For any elements* $a_{12}, b_{12} \in \mathscr{R}_{12}$, $b_{21}, c_{21} \in \mathscr{R}_{21}$ *and* $t_{22} \in \mathscr{R}_{22}$ *the following hold: (i)* $d_m(a_{12}t_{22} + b_{12}t_{22}) = d_m(a_{12}t_{22}) + d_m(b_{12}t_{22})$*; (ii)* $d_m(t_{22}b_{21} + t_{22}c_{21}) = d_m(t_{22}b_{21}) + d_m(t_{22}c_{21})$.

Proof. First, we observe that the following identity holds

$$a_{12}t_{22} + b_{12}t_{22} = \underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((a_{12} + t_{22}) \circ (b_{12} + t_{22}))) \cdots).$$

As a result, we have by Lemma 2.3(iii) that

$$\begin{aligned}
& d_m(a_{12}t_{22} + b_{12}t_{22}) \\
= \;& d_m(\underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((a_{12} + t_{22}) \circ (b_{12} + t_{22}))) \cdots)) \\
= \;& \underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ ((a_{12} + t_{22}) \circ d_m(b_{12} + t_{22}))) \cdots) \\
& + \underbrace{e_1 \circ (\cdots (e_1}_{(n-2)-\text{times}} \circ (d_m(a_{12} + t_{22}) \circ (b_{12} + t_{22}))) \cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_1\neq m,\; p_2\neq m}} \underbrace{d_{p_n}(e_1) \circ (\cdots (d_{p_3}(e_1)}_{(n-2)-\text{times}} \\
& \qquad\qquad \circ (d_{p_2}(a_{12} + t_{22}) \circ d_{p_1}(b_{12} + t_{22}))) \cdots) \\
= \;& e_1 \circ (\cdots (e_1 \circ (a_{12} \circ d_m(b_{12}))) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (a_{12} \circ d_m(t_{22}))) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (t_{22} \circ d_m(b_{12}))) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (t_{22} \circ d_m(t_{22}))) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (d_m(a_{12}) \circ b_{12})) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (d_m(a_{12}) \circ t_{22})) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (d_m(t_{22}) \circ b_{12})) \cdots) \\
& + e_1 \circ (\cdots (e_1 \circ (d_m(t_{22}) \circ t_{22})) \cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_1\neq m,\; p_2\neq m}} d_{p_n}(e_1) \circ (\cdots (d_{p_k}(e_1) \circ (\cdots (d_{p_2}(a_{12}) \circ d_{p_1}(b_{12})) \cdots)) \cdots)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{p_1+\ldots+p_n=m\\ p_1\neq m,\ p_2\neq m}} d_{p_n}(e_1)\circ(\cdots(d_{p_k}(e_1)\circ(\cdots(d_{p_2}(a_{12})\circ d_{p_1}(t_{22}))\cdots))\cdots)\\
&+ \sum_{\substack{p_1+\ldots+p_n=m\\ p_1\neq m,\ p_2\neq m}} d_{p_n}(e_1)\circ(\cdots(d_{p_k}(e_1)\circ(\cdots(d_{p_2}(t_{22})\circ d_{p_1}(b_{12}))\cdots))\cdots)\\
&+ \sum_{\substack{p_1+\ldots+p_n=m\\ p_1\neq m,\ p_2\neq m}} d_{p_n}(e_1)\circ(\cdots(d_{p_k}(e_1)\circ(\cdots(d_{p_2}(t_{22})\circ d_{p_1}(t_{22}))\cdots))\cdots)\\
=\ & d_m(e_1\circ(\cdots(e_1\circ(a_{12}\circ b_{12}))\cdots))+d_m(e_1\circ(\cdots(e_1\circ(a_{12}\circ t_{22}))\cdots))\\
&+d_m(e_1\circ(\cdots(e_1\circ(t_{22}\circ b_{12}))\cdots))+d_m(e_1\circ(\cdots(e_1\circ(t_{22}\circ t_{22}))\cdots))\\
=\ & d_m(a_{12}t_{22})+d_m(b_{12}t_{22}).
\end{aligned}
$$

Using a similar argument to the previous case we prove that $d_m(t_{22}b_{21}+t_{22}c_{21}) = d_m(t_{22}b_{21}) + d_m(t_{22}c_{21})$, from the identity

$$t_{22}b_{21}+t_{22}c_{21} = \underbrace{e_1\circ(\cdots(e_1}_{(n-2)-\text{times}}\circ((c_{21}+t_{22})\circ(b_{21}+t_{22})))\cdots).$$

□

Lemma 2.5. *For any elements $a_{12}, b_{12} \in \mathscr{R}_{12}$ and $b_{21}, c_{21} \in \mathscr{R}_{21}$ the following hold: (i) $d_m(a_{12}+b_{12}) = d_m(a_{12}) + d_m(b_{12})$; (ii) $d_m(b_{21}+c_{21}) = d_m(b_{21}) + d_m(c_{21})$.*

Proof. For any elements $r^k_{i_kj_k} \in \mathscr{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$, or $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, we have by Lemma 2.4(i) that

$$
\begin{aligned}
& d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots((a_{12}+b_{12})\circ r^1_{i_1j_1})\cdots))\\
=\ & d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(a_{12}\circ r^1_{i_1j_1})\cdots))\\
&+d_m(r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(b_{12}\circ r^1_{i_1j_1})\cdots))\\
=\ & r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(d_m(a_{12})\circ r^1_{i_1j_1})\cdots)\\
&+ \sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})
\end{aligned}
$$

$$\circ(\cdots(d_{p_2}(a_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)$$
$$+r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(d_m(b_{12})\circ r^1_{i_1j_1})\cdots)$$
$$+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots). \quad (5)$$

Considering $a_{11}=c_{21}=d_{22}=0$ and replacing b_{12} by $a_{12}+b_{12}$, in (2), and subtracting (2) from (5), it results

$$r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots((d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))\circ r^1_{i_1j_1})\cdots)=0. \quad (6)$$

As a consequence, if $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$ we get

$$r^1_{11}(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))r^2_{22}\cdots r^{n-1}_{22}$$
$$+r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))r^1_{11}=0,$$

by (6), which results that

$$r^1_{11}(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))r^2_{22}\cdots r^{n-1}_{22}$$
$$=r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))r^1_{11}=0.$$

It implies that $(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))_{12}=(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))_{21}=0$. If $(i_k,j_k)=(2,2)$ for all $(k=1,\cdots,n-1)$, then by (6), we obtain

$$r^{n-1}_{22}\circ(\cdots((d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))\circ r^1_{22})\cdots)=0.$$

which results $(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))_{22}=0$, by [2, Lema 2.1.(ii)]. If $(i_1,j_1)=(1,2)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, then by (6) and the previous cases, we conclude that

$$(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))r^1_{12}r^2_{22}\cdots r^{n-1}_{22}=0.$$

It therefore follows that $(d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12}))_{11}=0$. Consequently, $d_m(a_{12}+b_{12})-d_m(a_{12})-d_m(b_{12})=0$.

Similarly, we prove (ii). $\square$

Lemma 2.6. *For any elements $a_{11}, b_{11} \in \mathscr{R}_{11}$ and $c_{22}, d_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_m(a_{11}+b_{11}) = d_m(a_{11}) + d_m(b_{11})$; (ii) $d_m(c_{22}+d_{22}) = d_m(c_{22}) + d_m(d_{22})$.*

Proof. For any elements $r^k_{i_kj_k} \in \mathscr{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$, or $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, we have by Lemma 2.5(i) that

$$\begin{aligned}
& d_m(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots((a_{11}+b_{11}) \circ r^1_{i_1j_1})\cdots)) \\
= \; & d_m(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(a_{11} \circ r^1_{i_1j_1})\cdots)) \\
& + d_m(r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(b_{11} \circ r^1_{i_1j_1})\cdots)) \\
= \; & r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(d_m(a_{11}) \circ r^1_{i_1j_1})\cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}}) \circ (\cdots(d_{p_k}(r^k_{i_kj_k}) \\
& \qquad \circ(\cdots(d_{p_2}(a_{11}) \circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots) \\
& + r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots(d_m(b_{11}) \circ r^1_{i_1j_1})\cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}}) \circ (\cdots(d_{p_k}(r^k_{i_kj_k}) \\
& \qquad \circ(\cdots(d_{p_2}(b_{11}) \circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots).
\end{aligned} \tag{7}$$

Considering $b_{12} = c_{21} = d_{22} = 0$ and replacing a_{11} by $a_{11} + b_{11}$, in (2), and subtracting (2) from (7), we get

$$r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots((d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11})) \circ r^1_{i_1j_1})\cdots) = 0. \tag{8}$$

As a consequence, if $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, by (8) we obtain

$$\begin{aligned}
& r^1_{11}(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))r^2_{22}\cdots r^{n-1}_{22} \\
& \qquad = r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))r^1_{11} = 0.
\end{aligned}$$

which shows that $(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))_{12} = (d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))_{21} = 0$. If $(i_k, j_k) = (2,2)$ for all $(k = 1, \cdots, n-1)$, then by (8), we obtain

$$r_{22}^{n-1} \circ (\cdots((d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11})) \circ r_{22}^1)\cdots) = 0$$

resulting in $(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))_{22} = 0$, by [2, Lema 2.1.(ii)]. If $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, then by (8) yet and the previous cases, we obtain

$$(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))r_{12}^1 r_{22}^2 \cdots r_{22}^{n-1} = 0$$

which results that $(d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}))_{11} = 0$. Consequently, $d_m(a_{11}+b_{11}) - d_m(a_{11}) - d_m(b_{11}) = 0$.

Similarly, we prove (ii). $\square$

Lemma 2.7. *For any elements $b_{12} \in \mathscr{R}_{12}$ and $c_{21} \in \mathscr{R}_{21}$ the following holds $d_m(b_{12}+c_{21}) = d_m(b_{12}) + d_m(c_{21})$.*

Proof. For any elements $r_{i_kj_k}^k \in \mathscr{R}_{i_kj_k}$ $(k = 1, \cdots, n-1)$ with $(i_{n-1}, j_{n-1}) = (1,2)$ and $(i_k, j_k) = (1,1)$ $(k = 1, \cdots, n-2)$, or $(i_{n-1}, j_{n-1}) = (2,1)$ and $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-2)$, or $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (1,1)$ $(k = 2, \cdots, n-1)$, we have

$$\begin{aligned}
& d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots((b_{12}+c_{21}) \circ r_{i_1j_1}^1)\cdots)) \\
= \ & d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(b_{12} \circ r_{i_1j_1}^1)\cdots)) \\
& + d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(c_{21} \circ r_{i_1j_1}^1)\cdots)) \\
= \ & r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(d_m(b_{12}) \circ r_{i_1j_1}^1)\cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2\neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1}) \circ (\cdots(d_{p_k}(r_{i_kj_k}^k) \\
& \qquad \circ(\cdots(d_{p_2}(b_{12}) \circ d_{p_1}(r_{i_1j_1}^1))\cdots))\cdots) \\
& + r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(d_m(c_{21}) \circ r_{i_1j_1}^1)\cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2\neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1}) \circ (\cdots(d_{p_k}(r_{i_kj_k}^k) \\
& \qquad \circ(\cdots(d_{p_2}(c_{21}) \circ d_{p_1}(r_{i_1j_1}^1))\cdots))\cdots).
\end{aligned} \tag{9}$$

Considering $a_{11} = d_{22} = 0$ in (2) and subtracting (2) from (9) we get

$$r^{n-1}_{i_{n-1}j_{n-1}} \circ (\cdots((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) \circ r^1_{i_1j_1})\cdots) = 0. \quad (10)$$

As a consequence, if $(i_{n-1}, j_{n-1}) = (1,2)$ and $(i_k, j_k) = (1,1)$ $(k = 1, \cdots, n-2)$ we obtain

$$r^{n-1}_{12} \circ (r^{n-2}_{11} \circ (\cdots((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) \circ r^1_{11})\cdots)) = 0, \quad (11)$$

by (10). Multiplying (11) from right by t_{11}, then

$$r^{n-1}_{12}((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))r^1_{11}\cdots r^{n-2}_{11}t_{11} = 0$$

which results that $d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))_{21} = 0$. If $(i_{n-1}, j_{n-1}) = (2,1)$ and $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-2)$ we obtain

$$r^{n-1}_{21} \circ (r^{n-2}_{22} \circ (\cdots((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) \circ r^1_{22})\cdots)) = 0, \quad (12)$$

by (10) again. Multiplying (12) from right by t_{22}, then

$$r^{n-1}_{21}((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))r^1_{22}\cdots r^{n-2}_{22}t_{22} = 0$$

which yields that $(d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))_{12} = 0$. Now, let us prove that $(d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))_{11} = 0$. First, we observe that the following identity holds $b_{12}+c_{21} = \underbrace{e_1 \circ (\cdots((b_{12}+c_{21}) \circ e_1)}_{n-\text{times}}\cdots)$. As consequence, we have

$$\begin{aligned} d_m(b_{12}+c_{21}) = &= d_m(\underbrace{e_1 \circ (\cdots((b_{12}+c_{21}) \circ e_1)}_{n-\text{times}}\cdots)) \\ &= e_1 \circ (\cdots(d_m(b_{12}+c_{21}) \circ e_1)\cdots) \\ &+ \sum_{\substack{p_1+\ldots+p_n=m \\ p_2\neq m}} d_{p_n}(e_1) \circ (\cdots(d_{p_k}(e_1) \\ &\qquad \circ(\cdots(d_{p_2}(b_{12}+c_{21}) \circ d_{p_1}(e_1))\cdots))\cdots) \end{aligned}$$

$$
\begin{aligned}
&= e_1 \circ (\cdots (d_m(b_{12} + c_{21}) \circ e_1) \cdots) \\
&\quad + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2 \neq m}} d_{p_n}(e_1) \circ (\cdots (d_{p_k}(e_1) \\
&\qquad\qquad \circ(\cdots (d_{p_2}(b_{12}) \circ d_{p_1}(e_1)) \cdots)) \cdots) \\
&\quad + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2 \neq m}} d_{p_n}(e_1) \circ (\cdots (d_{p_k}(e_1) \\
&\qquad\qquad \circ(\cdots (d_{p_2}(c_{21}) \circ d_{p_1}(e_1)) \cdots)) \cdots). \qquad (13)
\end{aligned}
$$

Also, we observe that the identity holds $b_{12} = \underbrace{e_1 \circ (\cdots (b_{12} \circ e_1)}_{n-\text{times}} \cdots)$ which results in,

$$
\begin{aligned}
d_m(b_{12}) &= d_m(\underbrace{e_1 \circ (\cdots (b_{12} \circ e_1)}_{n-\text{times}} \cdots)) \\
&= e_1 \circ (\cdots (d_m(b_{12}) \circ e_1) \cdots) \\
&\quad + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2 \neq m}} d_{p_n}(e_1) \circ (\cdots (d_{p_k}(e_1) \\
&\qquad\qquad \circ(\cdots (d_{p_2}(b_{12}) \circ d_{p_1}(e_1)) \cdots)) \cdots). \qquad (14)
\end{aligned}
$$

Similarly, we have $c_{21} = \underbrace{e_1 \circ (\cdots (c_{21} \circ e_1)}_{n-\text{times}} \cdots)$ which yields

$$
\begin{aligned}
d_m(c_{21}) &= d_m(\underbrace{e_1 \circ (\cdots (c_{21} \circ e_1)}_{n-\text{times}} \cdots)) \\
&= e_1 \circ (\cdots (d_m(c_{21}) \circ e_1) \cdots) \\
&\quad + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2 \neq m}} d_{p_n}(e_1) \circ (\cdots (d_{p_k}(e_1) \\
&\qquad\qquad \circ(\cdots (d_{p_2}(c_{21}) \circ d_{p_1}(e_1)) \cdots)) \cdots). \qquad (15)
\end{aligned}
$$

It therefore follows that, subtracting (13) from (14) and (15) we obtain

$$
d_m(b_{12} + c_{21}) - d_m(b_{12}) - d_m(c_{21})
$$

$$= \quad e_1 \circ (\cdots((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) \circ e_1)\cdots). \qquad (16)$$

which implies that $(2^{n-1}-1)e_1(d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))e_1 = 0$. As results, we obtain $(d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))_{11} = 0$, since $\mathscr{R}$ is a $(2^{n-1}-1)$-torsion free ring. Finally, if $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (1,1)$ $(k = 2, \cdots, n-1)$, by (10) yet, we have

$$r_{11}^{n-1} \circ (r_{11}^{n-2} \circ (\cdots((d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) \circ r_{12}^1)\cdots)) = 0.$$

which shows that

$$r_{11}^{n-1} \cdots r_{11}^2 r_{12}^1 (d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21})) = 0. \qquad (17)$$

The identity (17) allows us to conclude that $(d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}))_{22} = 0$. Consequently, $d_m(b_{12}+c_{21}) - d_m(b_{12}) - d_m(c_{21}) = 0$. □

Lemma 2.8. *For any elements $a_{11} \in \mathscr{R}_{11}$, $b_{12} \in \mathscr{R}_{12}$, $c_{21} \in \mathscr{R}_{21}$ and $d_{22} \in \mathscr{R}_{22}$ the following hold: (i) $d_m(a_{11}+b_{12}+c_{21}) = d_m(a_{11}) + d_m(b_{12}) + d_m(c_{21})$; (ii) $d_m(b_{12}+c_{21}+d_{22}) = d_m(b_{12}) + d_m(c_{21}) + d_m(d_{22})$.*

Proof. For any elements $r_{i_k j_k}^k \in \mathscr{R}_{i_k j_k}$ $(k = 1, \cdots, n-1)$ with $(i_1, j_1) = (1,1)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, or $(i_k, j_k) = (2,2)$ $(k = 1, \cdots, n-1)$, or $(i_1, j_1) = (1,2)$ and $(i_k, j_k) = (2,2)$ $(k = 2, \cdots, n-1)$, we have by Lemma 2.7 that

$$\begin{aligned}
& d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots((a_{11}+b_{12}+c_{21}) \circ r_{i_1 j_1}^1)\cdots)) \\
= \;& d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(a_{11} \circ r_{i_1 j_1}^1)\cdots)) \\
& + d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(b_{12} \circ r_{i_1 j_1}^1)\cdots)) \\
& + d_m(r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(c_{21} \circ r_{i_1 j_1}^1)\cdots)) \\
= \;& r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(d_m(a_{11}) \circ r_{i_1 j_1}^1)\cdots) \\
& + \sum_{\substack{p_1+\ldots+p_n=m \\ p_2 \neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1}) \circ (\cdots(d_{p_k}(r_{i_k j_k}^k) \\
& \qquad \circ(\cdots(d_{p_2}(a_{11}) \circ d_{p_1}(r_{i_1 j_1}^1))\cdots))\cdots) \\
& + r_{i_{n-1}j_{n-1}}^{n-1} \circ (\cdots(d_m(b_{12}) \circ r_{i_1 j_1}^1)\cdots)
\end{aligned}$$

$$+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)$$
$$+r^{n-1}_{i_{n-1}j_{-1}}\circ(\cdots(d_m(c_{21})\circ r^1_{i_1j_1})\cdots)$$
$$+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})$$
$$\circ(\cdots(d_{p_2}(c_{21})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots). \qquad (18)$$

Considering $d_{22}=0$ in (2) and subtracting (2) from (18), we obtain

$$r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots((d_m(a_{11}+b_{12}+c_{21})$$
$$-d_m(a_{11})-d_m(b_{11})-d_m(c_{21}))\circ r^1_{i_1j_1})\cdots)=0. \qquad (19)$$

As a consequence, if $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, by (19) we get

$$r^1_{11}(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))r^2_{22}\cdots r^{n-1}_{22}$$
$$+r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))r^1_{11}=0$$

which implies that

$$r^1_{11}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))r^2_{22}\cdots r^{n-1}_{22}$$
$$=r^{n-1}_{22}\cdots r^2_{22}(d_m(a_{11}+b_{12})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))r^1_{11}=0.$$

It therefore follows that

$$(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))_{12}$$
$$=(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))_{21}=0.$$

If $(i_k,j_k)=(2,2)$ for all $(k=1,\cdots,n-1)$, then by (19), we get

$$r^{n-1}_{22}\circ(\cdots(d_m(a_{11}+b_{12}+c_{21})$$
$$-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))\circ r^1_{22})\cdots)=0.$$

which shows that $(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))_{22}=0$, by [2, Lema 2.1.(ii)]. If $(i_1,j_1)=(1,2)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, then by (19) we have

$$(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))r_{12}^1r_{22}^2\cdots r_{22}^{n-1}=0$$

which implies that $(d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21}))_{11}=0$. Consequently, $d_m(a_{11}+b_{12}+c_{21})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})=0$.

Similarly, we prove (ii). □

Lemma 2.9. *For any elements $a_{11}\in\mathscr{R}_{11}$, $b_{12}\in\mathscr{R}_{12}$, $c_{21}\in\mathscr{R}_{21}$ and $d_{22}\in\mathscr{R}_{22}$ holds $d_m(a_{11}+b_{12}+c_{21}+d_{22})=d_m(a_{11})+d_m(b_{12})+d_m(c_{21})+d_m(d_{22})$.*

Proof. For any elements $r_{i_kj_k}^k\in\mathscr{R}_{i_kj_k}$ $(k=1,\cdots,n-1)$ with $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, or $(i_k,j_k)=(1,1)$ $(k=1,\cdots,n-1)$, or $(i_k,j_k)=(2,2)$ $(k=1,\cdots,n-1)$, we have by Lemma 2.8 that

$$\begin{aligned}
& d_m(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots((a_{11}+b_{12}+c_{21}+d_{22})\circ r_{i_1j_1}^1)\cdots))\\
= {} & d_m(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(a_{11}\circ r_{i_1j_1}^1)\cdots))\\
& +d_m(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(b_{12}\circ r_{i_1j_1}^1)\cdots))\\
& +d_m(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(c_{21}\circ r_{i_1j_1}^1)\cdots))\\
& +d_m(r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(d_{22}\circ r_{i_1j_1}^1)\cdots))\\
= {} & r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(d_m(a_{11})\circ r_{i_1j_1}^1)\cdots)\\
& +\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1})\circ(\cdots(d_{p_k}(r_{i_kj_k}^k)\\
& \qquad\circ(\cdots(d_{p_2}(a_{11})\circ d_{p_1}(r_{i_1j_1}^1))\cdots))\cdots)\\
& +r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(d_m(b_{12})\circ r_{i_1j_1}^1)\cdots)\\
& +\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1})\circ(\cdots(d_{p_k}(r_{i_kj_k}^k)\\
& \qquad\circ(\cdots(d_{p_2}(b_{12})\circ d_{p_1}(r_{i_1j_1}^1))\cdots))\cdots)\\
& +r_{i_{n-1}j_{n-1}}^{n-1}\circ(\cdots(d_m(c_{21})\circ r_{i_1j_1}^1)\cdots)\\
& +\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r_{i_{n-1}j_{n-1}}^{n-1})\circ(\cdots(d_{p_k}(r_{i_kj_k}^k)
\end{aligned}$$

$$
\begin{aligned}
&\circ(\cdots(d_{p_2}(c_{21})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots)\\
&+r^{n-1}_{i_{n-1}j_{n-1}}\circ(\cdots(d_m(d_{22})\circ r^1_{i_1j_1})\cdots)\\
&+\sum_{\substack{p_1+\ldots+p_n=m\\ p_2\neq m}} d_{p_n}(r^{n-1}_{i_{n-1}j_{n-1}})\circ(\cdots(d_{p_k}(r^k_{i_kj_k})\\
&\circ(\cdots(d_{p_2}(d_{22})\circ d_{p_1}(r^1_{i_1j_1}))\cdots))\cdots).
\end{aligned}\tag{20}
$$

Subtracting (2) from (20) it results that

$$
\begin{aligned}
r^{n-1}_{i_{n-1}j_{n-1}}&\circ(\cdots((d_m(a_{11}+b_{12}+c_{21}+d_{22})\\
&-d_m(a_{11})-d_m(b_{11})-d_m(c_{21})-d_m(d_{22}))\circ r^1_{i_1j_1})\cdots)=0.
\end{aligned}\tag{21}
$$

As a consequence, if $(i_1,j_1)=(1,1)$ and $(i_k,j_k)=(2,2)$ $(k=2,\cdots,n-1)$, we get $(d_m(a_{11}+b_{12}+c_{21}+d_{22})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})-d_m(d_{22}))_{12}=0$ and $(d_m(a_{11}+b_{12}+c_{21}+d_{22})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})-d_m(d_{22}))_{21}=0$, by (21). If $(i_k,j_k)=(1,1)$ for all $(k=1,\cdots,n-1)$, then $(d_m(a_{11}+b_{12}+c_{21}+d_{22})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})-d_m(d_{22}))_{11}=0$, by (21) and [2, Lemma 2.1.(ii)]. If $(i_k,j_k)=(2,2)$ for all $(k=1,\cdots,n-1)$, we have $(d_m(a_{11}+b_{12}+c_{21}+d_{22})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})-d_m(d_{22}))_{22}=0$, by (21) and [2, Lemma 2.1.(ii)] again. It therefore follows that $d_m(a_{11}+b_{12}+c_{21}+d_{22})-d_m(a_{11})-d_m(b_{12})-d_m(c_{21})-d_m(d_{22})=0$. □

Lemma 2.10. *For any elements $a,b\in\mathscr{R}$ holds $d_m(a+b)=d_m(a)+d_m(b)$.*

Proof. The proof is a direct consequence of Lemmas 2.5, 2.6 and 2.9. Thus, d_m is additive. □

Now we are able to prove the Theorem 2.1.

Proof of Theorem 2.1. The Lemma 2.10 and the second principle of mathematical induction allows us to conclude that d_m is an additive map, for each non-negative integer m. Consequently, $D=\{d_m\}_{m\in\mathbb{N}}$ is a family of additive maps. The prove is complete. □

We may therefore state the following corollaries.

Corollary 2.1. *Let $\mathscr{R}$ be a ring* 2 *and* $(2^{n-1}-1)$*-torsion free prime ring containing a non-trivial idempotent* e_1 *and satisfying the following condition:* $e_2r_1e_2 \circ (\cdots (e_2ae_2 \circ e_2r_1e_2)\cdots) = 0$, *for all* $r_1 \in \mathscr{R}$, *implies* $e_2ae_2 = 0$. *Then every Jordan* n*-tuple higher derivable map* $D = \{d_m\}_{m\in\mathbb{N}}$ *is additive.*

As a direct consequence of the above Corollary 2.1 we have the following corollary.

Corollary 2.2. *Let $\mathscr{X}$ be a Banach space with* $\dim \mathscr{X} > 1$ *and* $\mathscr{A} \subset \mathscr{B}(\mathscr{X})$ *a standard operator algebra. Then every Jordan* n*-tuple higher derivable map* $D = \{d_m\}_{m\in\mathbb{N}}$ *is additive.*

Proof. Since it is well known that $\mathscr{A}$ is a prime ring of characteristic zero and containing a non-trivial idempotent, then it is sufficient to prove that the condition in Corollary 2.1 is satisfied. Hence, since $\mathscr{A}$ is dense in $\mathscr{B}(\mathscr{X})$ under the strong operator topology, let us consider a net $\{r_\lambda\}_{\lambda\in\Lambda} \subset \mathscr{B}(\mathscr{X})$ such that $SOT-\lim_\lambda r_\lambda = 1$. The limit in $e_2r_\lambda e_2 \circ (\cdots (e_2ae_2 \circ e_2r_\lambda e_2)\cdots) = 0$, for all $r_\lambda \in \mathscr{A}$, leads us to conclude that $e_2ae_2 = 0$. □

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