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HAMILTON-JACOBI TREATMENT OF SUPERSTRING AND QUANTIZATION OF FIELDS WITH CONSTRAINTS

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Abstract

The Hamilton-Jacobi formalism of constrained systems is used to study superstring. That obtained the equations of motion for a singular system as total differential equations in many variables. These equations of motion are in exact agreement with those equations obtained using Dirac's method. Moreover, the Hamilton-Jacobi quantization of a constrained system is discussed. Quantization of the relativistic local free field with linear velocity of dimension D containing second-class constraints is studied. The set of Hamilton-Jacobi partial differential equations and the path integral of these theories are obtained by using the canonical path integral quantization. We figured out that the Hamilton-Jacobi path integral quantization of this system is in exact agreement with that given by using Senjanovic method. Furthermore, Hamilton-Jacobi path integral quantization of the scalar field coupled to two flavours of fermions through Yukawa couplings is obtained directly as an integration over the canonical phase space. Hamilton-Jacobi quantization is applied to the constraint field systems with finite degrees of freedom by investigating the integrability conditions without using any gauge fixing condition.

keywords: Field Theory; Gauge Fields; Hamilton-Jacobi Formulation; Singular Lagrangian, Path Integral Quantization of Constrained Systems.

1 Introduction

The generalized Hamiltonian dynamics describing systems with constraints was initiated by Dirac [1, 2], who established a formalism for treating constraint singular systems. The presence of constraints in such theories requires care when applying Dirac's method, especially when first-class constraints arise since the first-class constraints are generators of gauge transformations which lead to the gauge freedom. Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: so-called first-class and second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are by definition second-class. Most physicists believe that this distinction is quite important not only in classical theories but also in quantum mechanics [3, 4].

As a first step in the present work, we intend to study a singular system with Lagrangian describing superstring from the point of view of the Hamilton-Jacobi formalism which has been developed by Güler [5, 6] to investigate constrained systems. The equivalent Lagrangian method [7] is used to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE). The study of such systems through Dirac's generalized Hamiltonian formalism has already been extensively developed in literature [3, 4] to investigate theoretical models in contemporary elementary particle physics and will be used here for comparative purposes.

Despite the success of Dirac's approach in studying singular systems, which is demonstrated by the wide number of physical systems to which this formalism has been applied, it is instructive to study singular systems through other formalisms, since different procedures will provide different views for the same problems, even for nonsingular systems. The Hamilton-Jacobi approach that we study in this work, is applied to some physical examples [8, 9, 10, 11, 12, 13]. But it is still lacking to a better understanding of this approach utility in the studying singular systems, and such understanding can only be achieved through its application to other interesting physical systems. From our aims in this work is to treat the superstring constraint system by the Hamilton-Jacobi approach and compare the results to those obtained through Dirac's method.

In the case of unconstrained systems, the Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics. The first study of the Hamilton-Jacobi equations for arbitrary first-order actions was initiated by Santilli [14]. The quantization and construction of functional integral for theories with first-class constraints in canonical gauge was given by Faddeev and Popov [15, 16]. Faddeev's method is generalized by Senjanovic [17] to the case of appearance of the second-class constraints in the theory. Moreover, Fradkin [18] considered quantization of bosonic theories with first- and second-class constraints and the extension to include in such gauges. Gitman and Tyutin [3] discussed the Hamiltonian formalism of gauge theories in an arbitrary gauge and the canonical quantization of singular theories. In the Hamiltonian-Jacobi approach, the distinction between first- and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables, which need to investigate the integrability conditions. In other words, the integrability conditions may lead to new constraints. Moreover, it is shown that gauge fixing, which is a basic procedure to study singular systems by Dirac's method, is not necessary if the canonical method is used [7]. The path integral formulation based on the canonical method is obtained in Refs. [19, 20, 21, 22].

In Ref. [23], We have studied successfully the first-class constraints in canonical gauge by applying the Faddeev and Hamilton-Jacobi methods to obtain the path integral quantization of the scalar field coupled minimally to the vector potential. That led to the same results by the two methods which prove that the Hamilton-Jacobi method apple to quantized the first-class constraints. For more complement of confirmation the successful of Hamilton-Jacobi method, we quantize the relativistic local free field with a linear velocity of dimension D with the second-class constraints. The path integral quantization of this field is obtained by using the Senjanovic and Hamilton-Jacobi methods. We noticed that Faddeev [23], Popov and Senjanovic treatment need gauge-fixing conditions to obtain the path integral over the canonical variable, which is not always an easy task. However, if the Hamilton-Jacobi approach [5, 6] is used, the gauge fixing is not necessary to analyze singular systems [9]. From the previous comparison, we figure out the ability and simplicity of using the Hamilton-Jacobi approach for studying the constraint systems. So, in the end, we get the path integral quantization of the scalar field coupled to two flavours of fermions through Yukawa couplings by using Hamilton-Jacobi quantization.

2 Hamilton-Jacobi Approach

In this section, we approach the constrained systems by Hamilton-Jacobi treatment, which solves the gauge-fixing problem naturally.
Güler [5, 6] has developed a completely different method to investigate singular systems. He started with the Hess matrix elements A_{ik} of second derivatives of the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_i, \partial\varphi_i, \tau)$, $i = 1, ..., n$, which defined as

$$A_{ik} = \frac{\delta^2 \mathcal{L}(\varphi_i, \partial\varphi_i, \tau)}{\delta(\partial\varphi_i)\delta(\partial\varphi_k)}, \quad i, k = 1, 2, ..., n, \tag{1}$$

of rank $(n - r)$, $r < n$, with dependent momenta r. The equivalent Lagrangian method [7] is used to obtain the set of Hamilton-Jacobi Partial Differential Equations (HJPDE). The generalized momenta corresponding to generalized coordinates φ_i are defined as

$$\pi_a = \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_a)}, \qquad a = 1, 2, \ldots, n - r, \tag{2}$$

$$\pi_j = \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_j)}, \qquad j = n - r + 1, \ldots, n, \tag{3}$$

where φ_i are divided into two sets φ_a and x_j. Since the rank of Hess matrix is $(n - r)$, one may solve Eq.(2) for $\partial_\mu \varphi_a$ as

$$\partial_\mu \varphi_a = \partial_\mu \varphi_a(\varphi_i, \pi_a, \partial_\mu \varphi_j; \chi_\mu) \equiv \omega_a, \tag{4}$$

By substituting Eq. (4) into Eq. (3), we get

$$\pi_j = \frac{\overrightarrow{\delta} \mathcal{L}}{\delta(\partial_\mu \varphi_j)}\bigg|_{\partial_\mu \varphi_a = \omega_a} \equiv -\mathcal{H}_j(\varphi_i, \partial_\mu \varphi_\nu, \pi_a; \chi_\mu), \tag{5}$$

which indicate to the fact that the generalized momenta π_j depend on π_a. That is a normal result of the singular nature of the Lagrangian.
The canonical Hamiltonian $\mathcal{H}_0$ is given by the following definition

$$\mathcal{H}_0 = -\mathcal{L}(\varphi_i, \partial_\mu \varphi_\nu, \partial_\mu \varphi_a \equiv \omega_a, \chi_\mu) + \pi_a \omega_a + \pi_j \partial_\mu \varphi_j\big|_{\pi_\nu = -\mathcal{H}_\nu}. \tag{6}$$

The expression of the set of Hamilton-Jacobi Partial Differential Equations (HJPDE) is

$$\mathcal{H}_0'\left(\tau,\ \varphi_\nu,\ \varphi_a,\ \pi_i = \frac{\overrightarrow{\delta} S}{\delta \varphi_i},\ \pi_0 = \frac{\overrightarrow{\delta} S}{\delta \chi_\mu}\right) = 0, \tag{7}$$

$$\mathcal{H}'_\mu\left(\tau,\ \varphi_\nu,\ \varphi_a,\ \pi_i = \frac{\overrightarrow{\delta} S}{\delta\varphi_i},\ \pi_0 = \frac{\overrightarrow{\delta} S}{\delta\chi_\mu}\right) = 0, \tag{8}$$

where S being the action.
Eqs. (7) and (8) may be expressed in a compact form as

$$\mathcal{H}'_\alpha\left(\tau,\ \varphi_\nu,\ \varphi_a,\ \pi_i = \frac{\overrightarrow{\delta} S}{\delta\varphi_i},\ \pi_0 = \frac{\overrightarrow{\delta} S}{\delta\chi_\alpha}\right) = 0, \tag{9}$$

$$\alpha = 0, n-r+1, \ldots, n.$$

where

$$\mathcal{H}'_0 = \pi_0 + \mathcal{H}_0 = 0\,, \tag{10}$$

$$\mathcal{H}'_\mu = \pi_j + \mathcal{H}_j = 0\,. \tag{11}$$

Here $\mathcal{H}'_0$ can be interpreted as the generator of time evolution while $\mathcal{H}'_j$ are the generators of gauge transformation.
The fundamental equations of the equivalent Lagrangian method are

$$\pi_0 = \frac{\overrightarrow{\delta} S}{\overrightarrow{\delta}\chi_\mu} \equiv -\mathcal{H}_0(\varphi_i, \delta_\mu\varphi_\nu, \pi_a; \chi_\mu)\,, \tag{12}$$

$$\pi_a = \frac{\overrightarrow{\delta} S}{\delta\varphi_a}, \qquad \pi_j = \frac{\overrightarrow{\delta} S}{\delta\varphi_j} \equiv -\mathcal{H}_j\,, \tag{13}$$

with $\varphi_0 = \chi_\mu$. That gives the equations of motion as total differential equations in many variables as

$$d\varphi_r = \frac{\overrightarrow{\delta}\mathcal{H}'_\alpha}{\delta\pi_r}\,d\chi_\alpha, \qquad r = 0, 1, \ldots, n\,, \tag{14}$$

$$d\pi_a = -\frac{\overrightarrow{\delta}\mathcal{H}'_\alpha}{\delta\varphi_a}\,d\chi_\alpha, \qquad a = 1, \ldots, n-r\,, \tag{15}$$

$$d\pi_\mu = -\frac{\overrightarrow{\delta}\mathcal{H}'_\alpha}{\delta\varphi_\mu}\,d\chi_\alpha, \qquad \mu = n-r+1, \ldots, n\,, \tag{16}$$

$$dZ = \left(-\mathcal{H}_\alpha + \pi_a\frac{\overrightarrow{\delta}\mathcal{H}'_\alpha}{\delta\pi_a}\right)\,d\chi_\alpha, \quad \alpha = 0, n-r+1, \ldots, n\,, \tag{17}$$

where $Z = S(\chi_\alpha, \varphi_a)$. These equations are integrable if and only if

$$d\mathcal{H}'_0 = 0\,, \tag{18}$$
$$d\mathcal{H}'_\beta = 0, \qquad \beta = 1, 2, \ldots, r\,. \tag{19}$$

In the case of not satisfied the conditions (18) and (19) identically, one has to consider them as new constraints and then we examine again the variations of them. One is repeating this procedure until obtain a set of conditions with all variations vanish.

The investigation of the integrability conditions [24, 25] can be also done by using the operator method, where the linear operators X_α corresponding to the set (14 - 16) are defined as

$$X_\alpha f(\chi_\beta, \varphi_a, \pi_a, z) = \frac{\delta f}{\delta \chi_\alpha} + \frac{\delta \mathcal{H}'_\alpha}{\delta \pi_a}\,\frac{\delta f}{\delta \varphi_a} - \frac{\delta \mathcal{H}'_\alpha}{\delta \varphi_a}\,\frac{\delta f}{\delta \pi_a} + \left(- \mathcal{H}_\alpha + \pi_a \frac{\delta \mathcal{H}'_\alpha}{\delta \pi_a}\right)\frac{\delta f}{\delta z}\,. \tag{20}$$

The system is integrable, if the bracket relations

$$[X_\alpha, X_\beta]f = (X_\alpha X_\beta - X_\beta X_\alpha)f = C^\gamma_{\alpha\beta} X_\gamma f; \quad \forall\ \alpha, \beta, \gamma = 0, n-r+1, \ldots, n\,, \tag{21}$$

are hold. If the relations (14 - 16) are not satisfied identically, we add the bracket relations, which cannot be expressed in this form as new operators. So the numbers of independent operators are increased, and a new complete system can be obtained. Then the new operators can be written in the Jacobi form, and we fined the corresponding integrable system of the total differential equations.

3 Path Integral Quantization

In this section, we briefly review the Senjanovic's and Hamilton-Jacobi methods for studying the path integral quantization of constrained systems.

3.1 Senjanovic Method

We generalize Faddeeve's method [15] to the case when second-class constraints are present. This generalization is called Senjanovic's method.
Consider a mechanical system with α first-class constraints ϕ_a, β second-class constraints θ_b, and the gauge conditions associated with the first-class

constraints χ_a. Let the χ_a be chosen in such a way that $\{\chi_a, \chi_b\} = 0$. Then the expression for the S-matrix element is [17]

$$\langle Out \mid S \mid In \rangle = \int exp\left[i\int_{-\infty}^{\infty}(p_i\dot{q}_i - H_0)\,dt\right]\prod_t d\mu(q(t), p(t))\,, \qquad (22)$$

and

$$d\mu(q,p) = \left(\prod_{a=1}^{\alpha}\delta(\chi_a)\delta(\phi_a)\right) det||\{\chi_a, \phi_a\}|| \\ \times \prod_{b=1}^{\beta}\delta(\theta_b)\, det||\{\theta_a, \theta_b\}||^{\frac{1}{2}}\prod_{i=1}^{n}dp_i\, dq^i\,. \quad (23)$$

where H_0 is the Hamiltonian of the system and $d\mu(q,p)$ is the measure of integration.

3.2 Hamilton-Jacobi Quantization

In Refs. [19, 20, 21, 23, 24, 25, 26], the path integral formulation of the constrained systems is studied. For computing the Hamilton-Jacobi path integral, one has to consider a singular Lagrangian as seen in section II. The canonical Hamiltonian H_0 defined in Eq. (6), and the set of HJPDE is expressed in Eqs. (7) and (8). As we define $p_\beta = \frac{\partial S[q_a;x_\alpha]}{\partial x_\beta}$ and $p_a = \frac{\partial S[q_a;x_\alpha]}{\partial q_a}$ with $x_0 = t$ and S being the action. The total differential equations given in (14 - 17) are integrable if (18) and (19) are hold [26]. If conditions (18) and (19) are not satisfied identically, one considers them as new constraints and again consider their variations.

Thus, repeating this procedure one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide to obtain the set of canonical phase space coordinates (q_a, p_a) as functions of t_α, besides the canonical action integral is obtained in terms of the canonical coordinates. H'_α can be interpreted as the infinitesimal generator of canonical transformations given by parameters

t_α. In this case path integral representation may be written as

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-p} dq^a dp^a exp \left\{ \int_{t_\alpha}^{t'_\alpha} \left(-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha \right\},$$
$$a = 1, \ldots, n-p, \qquad \alpha = 0, n-p+1, \ldots, n. \quad (24)$$

In fact, this path integral is an integration over the canonical phase-space coordinates (q^a, p^a).

4 Hamilton-Jacobi treatment of superstring

In this section we treat supersting constraint system by Dirac's method and then apply Hamilton-Jacobi method.

4.1 Dirac's formulation of superstring

Consider a Lagrangian describes a superstring system

$$L = -\frac{1}{2\pi} \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) - \jmath^\alpha A_\alpha - \frac{1}{4\pi} F^{\alpha\beta} F_{\alpha\beta}, \qquad (25)$$

Where A^α is a world sheet potential analogous to the electromagnetic potential. The world sheet current density

$$\jmath_\alpha = \frac{1}{2\pi} q \overline{\psi}^\mu \rho^\alpha \psi_\mu, \qquad (26)$$

acts as a source for the gauge field (A^α), and the electromagnetic tensor is defined as $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.

The Lagrangian (25) is singular, since the rank of the Hess matrix (1) is four. The generalized momenta (2) and (3) can be written as

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{1}{\pi} \partial^0 X_\mu, \qquad (27)$$

$$p_{\overline{\psi}}^\mu = \frac{\partial L}{\partial \dot{\overline{\psi}}^\mu} = 0 = -H_{\overline{\psi}}, \qquad (28)$$

$$p_\psi^\mu = \frac{\partial L}{\partial \dot{\psi}_\mu} = \frac{i}{2\pi} \overline{\psi}^\mu \rho^0 = -H_\psi, \qquad (29)$$

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = \frac{1}{\pi} F^{i0} , \tag{30}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0 = -H_1 . \tag{31}$$

Equations (27) and (30), respectively leads us to express the velocities $\dot{X}^\mu$ and $\dot{A}_i$ as

$$\dot{X}^\mu = -\pi \, p^\mu , \tag{32}$$

$$\dot{A}_i = -\pi \, \pi_i + \partial_i A_0 . \tag{33}$$

The Hamiltonian density is given by

$$\begin{aligned} H_0 = &- \frac{\pi}{2}(p^\mu \, p_\mu + \pi^i \, \pi_i) + \pi^i \, \partial_i A_0 \\ &+ \frac{1}{2\pi}[\partial_i X^\mu \, \partial^i X_\mu - i\overline{\psi}^\mu \rho^i \, \partial_i \psi_\mu + q\overline{\psi}^\mu \rho^\alpha \psi_\mu A_\alpha + \frac{1}{2} F^{ij} \, F_{ij}] . \end{aligned} \tag{34}$$

The total Hamiltonian density is constructed as

$$\begin{aligned} H_T = &- \frac{\pi}{2}(p^\mu \, p_\mu + \pi^i \, \pi_i) + \pi^i \, \partial_i A_0 + \frac{1}{2\pi}[\partial_i X^\mu \, \partial^i X_\mu - i\overline{\psi}^\mu (\rho^i \, \partial_i \\ &+ iq\rho^\alpha A_\alpha)\psi_\mu + \frac{1}{2} F^{ij} \, F_{ij}] + \lambda_{\overline{\psi}} p^\mu_{\overline{\psi}} + \lambda_\psi (p^\mu_\psi - \frac{i}{2\pi} \overline{\psi}^\mu \rho^0) + \lambda_1 \pi_0 , \end{aligned} \tag{35}$$

where $\lambda_{\overline{\psi}}$, λ_ψ and λ_1 are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

$$\dot{H'}_{\overline{\psi}} = \{H'_{\overline{\psi}}, H_T\} = \frac{1}{2\pi}(i\rho^i \partial_i - q\rho^\alpha \, A_\alpha)\psi_\mu + \frac{i}{2\pi}\rho^0 \, \lambda_\psi \approx 0 , \tag{36}$$

$$\dot{H'}_\psi = \{H'_\psi , H_T\} = -\frac{1}{2\pi}\overline{\psi}^\mu (i\overleftarrow{\partial_i}\rho^i + q\rho^\alpha \, A_\alpha) - \frac{i}{2\pi}\lambda_{\overline{\psi}}\rho^0 \approx 0 , \tag{37}$$

$$\dot{H'}_1 = \{H'_1, H_T\} = \partial_i \pi^i - \frac{1}{2\pi} \, q\overline{\psi}^\mu \rho^0 \psi_\mu \approx 0 . \tag{38}$$

Relations (36) and (37) fix the multipliers λ_ψ and $\lambda_{\overline{\psi}}$ respectively as

$$\lambda_\psi = -(\rho^0 \, \rho^i \, \partial_i + iq \, \rho^0 \, \rho^\alpha \, A_\alpha) \, \psi_\mu , \tag{39}$$

$$\lambda_{\overline{\psi}} = -\overline{\psi}^\mu (\overleftarrow{\partial_i} \, \rho^i \, \rho^0 - iq \, \rho^\alpha \, \rho^0 \, A_\alpha) . \tag{40}$$

Eq. (38) lead to the secondary constraints

$$H_1'' = \partial_i \pi^i - \frac{1}{2\pi}\, q\overline{\psi}^{\,\mu} \rho^0 \psi_\mu \approx 0\,. \tag{41}$$

There are no tertiary constraints, since

$$\dot{H''}_1 = \{H_1'', H_T\} = 0\,. \tag{42}$$

By taking suitable linear combinations of constraints, one has to find the first-class, that is

$$\Phi_1 = H_1' = \pi_0\,, \tag{43}$$

whereas the constraints

$$\Phi_2 = H_{\overline{\psi}}' = p_{\overline{\psi}}^{\mu}\,, \tag{44}$$

$$\Phi_3 = H_\psi' = p_\psi^\mu - \frac{i}{2\pi}\,\overline{\psi}^{\,\mu} \rho^0\,, \tag{45}$$

$$\Phi_4 = H_1'' = \partial_i \pi^i - \frac{1}{2\pi}\, q\overline{\psi}^{\,\mu} \rho^0 \psi_\mu\,, \tag{46}$$

are second-class.
The equations of motion read as

$$\dot{X}^\mu = \{X^\mu, H_T\} = -\pi\, p^\mu\,, \tag{47}$$

$$\dot{\overline{\psi}}^{\,\mu} = \{\overline{\psi}^{\,\mu}, H_T\} = \lambda_{\overline{\psi}}\,, \tag{48}$$

$$\dot{\psi}^{\,\mu} = \{\psi_\mu, H_T\} = \lambda_\psi\,, \tag{49}$$

$$\dot{A}^i = \{A^i, H_T\} = -(\pi\,\pi^i - \partial_i A_0)\,, \tag{50}$$

$$\dot{A}^0 = \{A^0, H_T\} = \lambda_1\,, \tag{51}$$

$$\dot{p}_\mu = \{p_\mu, H_T\} = \frac{1}{\pi}\,\partial^i \partial_i X^\mu\,, \tag{52}$$

$$\dot{p}_{\overline{\psi}}^{\mu} = \{p_{\overline{\psi}}^{\mu}, H_T\} = \frac{1}{2\pi}\left[(i\rho^i\,\partial_i - q\rho^\alpha\, A_\alpha)\psi_\mu + i\lambda_\psi\,\rho^0\right], \tag{53}$$

$$\dot{p}^{\mu}_{\psi} = \{p^{\mu}_{\psi}, H_T\} = -\frac{i}{2\pi}\,\overline{\psi}^{\,\mu}(\overleftarrow{\partial_i}\rho^i + iq\,\rho^{\alpha}\,A_{\alpha})\,, \tag{54}$$

$$\dot{\pi}^i = \{\pi^i, H_T\} = \frac{1}{2\pi}\,(2\partial^l F_{il} - q\,\overline{\psi}^{\,\mu}\,\rho^i\,\psi_{\mu})\,, \tag{55}$$

$$\dot{\pi}^0 = \{\pi^0, H_T\} = \partial_i\pi^i - \frac{1}{2\pi}\,q\,\overline{\psi}^{\,\mu}\,\rho^0\,\psi_{\mu}\,. \tag{56}$$

Substituting from Eq. (40) into Eq. (48), we get

$$i\overline{\psi}^{\,\mu}\rho^{\alpha}(\overleftarrow{\partial_{\alpha}} - iqA_{\alpha}) = 0\,, \tag{57}$$

and from Eq. (39) into Eqs. (49) and (53), we have

$$i(\partial_{\alpha} + iqA_{\alpha})\rho^{\alpha}\,\psi_{\mu} = 0\,, \tag{58}$$

$$\dot{p}^{\mu}_{\overline{\psi}} = 0\,. \tag{59}$$

We will contact oureselves with a partial gauge fixing by introducing gauge constraints for the first class primary constraints only, just to fix the multiplier λ_1 in Eq. (35). Since π^0 is vanishing weakly, a gauge choice near at hand would be

$$\Phi'_1 = A_0 = 0\,. \tag{60}$$

But for this forbids dynamics at all, since the requirement $\dot{A}_0 = 0$ implies $\lambda_1 = 0$.

In the following section the same system will be discussed using Hamilton-Jacobi approach.

4.2 Hamilton-Jacobi formulation of superstring

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (7) read as

$$H'_0 = p_0 + H_0 = 0\,, \tag{61}$$

$$H'_{\overline{\psi}} = p^{\mu}_{\overline{\psi}} + H_{\overline{\psi}} = p^{\mu}_{\overline{\psi}} = 0\,, \tag{62}$$

$$H'_{\psi} = p^{\mu}_{\psi} + H_{\psi} = p^{\mu}_{\psi} - \frac{i}{2\pi}\,\overline{\psi}^{\,\mu}\rho^0 = 0\,, \tag{63}$$

$$H_1' = \pi_0 + \Pi_1 = \pi_0 = 0\,. \tag{64}$$

The equations of motion are obtained as total differential equations follows:

$$\begin{aligned} dX^\mu &= \frac{\partial H_0'}{\partial p_\mu}\,dt + \frac{\partial H_{\overline{\psi}}'}{\partial p_\mu}\,d\overline{\psi}^{\,\mu} + \frac{\partial H_\psi'}{\partial p_\mu}\,d\psi_\mu + \frac{\partial H_1'}{\partial p_\mu}\,dA^0 \\ &= -\pi\,p^\mu\,dt\,, \end{aligned} \tag{65}$$

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\,dt + \frac{\partial H_{\overline{\psi}}'}{\partial \pi_i}\,d\overline{\psi}^{\,\mu} + \frac{\partial H_\psi'}{\partial \pi_i}\,d\psi_\mu + \frac{\partial H_1'}{\partial \pi_i}\,dA^0 \\ &= -(\pi\,\pi^i - \partial_i A_0)\,dt\,, \end{aligned} \tag{66}$$

$$\begin{aligned} dp_\mu &= -\frac{\partial H_0'}{\partial X_\mu}\,dt - \frac{\partial H_{\overline{\psi}}'}{\partial X_\mu}\,d\overline{\psi}^{\,\mu} - \frac{\partial H_\psi'}{\partial X_\mu}\,d\psi_\mu - \frac{\partial H_1'}{\partial X_\mu}\,dA^0 \\ &= \frac{1}{\pi}\,\partial^i\partial_i X^\mu\,dt\,, \end{aligned} \tag{67}$$

$$\begin{aligned} dp_{\overline{\psi}}^\mu &= -\frac{\partial H_0'}{\partial \overline{\psi}^{\,\mu}}\,dt - \frac{\partial H_{\overline{\psi}}'}{\partial \overline{\psi}^{\,\mu}}\,d\overline{\psi}^{\,\mu} - \frac{\partial H_\psi'}{\partial \overline{\psi}^{\,\mu}}\,d\psi_\mu - \frac{\partial H_1'}{\partial \overline{\psi}^{\,\mu}}\,dA^0 \\ &= \frac{1}{2\pi}\,(i\rho^i\,\partial_i - q\rho^\alpha\,A_\alpha)\psi_\mu\,dt + \frac{i}{2\pi}\,\rho^0 d\psi_\mu\,, \end{aligned} \tag{68}$$

$$\begin{aligned} dp_\psi^\mu &= -\frac{\partial H_0'}{\partial \psi_\mu}\,dt - \frac{\partial H_{\overline{\psi}}'}{\partial \psi_\mu}\,d\overline{\psi}^{\,\mu} - \frac{\partial H_\psi'}{\partial \psi_\mu}\,d\psi_\mu - \frac{\partial H_1'}{\partial \psi_\mu}\,dA^0 \\ &= -\frac{1}{2\pi}(i\,\partial_i\,\overline{\psi}^{\,\mu}\,\rho^i + q\,\overline{\psi}^{\,\mu}\,\rho^\alpha\,A_\alpha)\,dt\,, \end{aligned} \tag{69}$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H_0'}{\partial A_i}\,dt - \frac{\partial H_{\overline{\psi}}'}{\partial A_i}\,d\overline{\psi}^{\,\mu} - \frac{\partial H_\psi'}{\partial A_i}\,d\psi_\mu - \frac{\partial H_1'}{\partial A_i}\,dA^0 \\ &= \frac{1}{2\pi}\,(2\partial^l F_{il} - q\,\overline{\psi}^{\,\mu}\,\rho^i\,\psi_\mu)\,dt\,, \end{aligned} \tag{70}$$

$$d\pi^0 = -\frac{\partial H_0'}{\partial A_0}\,dt - \frac{\partial H_{\overline{\psi}}'}{\partial A_0}\,d\overline{\psi}^{\mu} - \frac{\partial H_{\psi}'}{\partial A_0}\,d\psi_{\mu} - \frac{\partial H_1'}{\partial A_0}\,dA^0$$
$$= (\partial_i \pi^i - \frac{1}{2\pi}\,q\,\overline{\psi}^{\mu}\,\rho^0\,\psi_{\mu})dt\,. \tag{71}$$

The integrability conditions imply that the variation of the constraints $H_{\overline{\psi}}'$, H_{ψ}' and H_1' should be identically zero; that is

$$dH_{\overline{\psi}}' = dp_{\overline{\psi}}^{\mu} = 0\,, \tag{72}$$

$$dH_{\psi}' = dp_{\psi}^{\mu} - \frac{i}{2\pi}\,d\overline{\psi}^{\mu}\rho^0 = 0\,, \tag{73}$$

$$dH_1' = d\pi_0 = 0\,. \tag{74}$$

The vanishing of total differential of H_1' leads to a new constraints

$$H_1'' = \partial_i \pi^i - \frac{1}{2\pi}\,q\overline{\psi}^{\mu}\rho^0\psi_{\mu}\,. \tag{75}$$

When we taking a gain the total differential of H_1'', we notice that it vanishes identically,

$$dH_1'' = 0\,. \tag{76}$$

From Eqs. (65) and (66), respectively we obtain

$$\dot{X}^{\mu} = -\pi\,p^{\mu}\,, \tag{77}$$

and

$$\dot{A}^i = -(\pi\,\pi^i - \partial_i A_0)\,. \tag{78}$$

Substituting from Eqs. (68) and (69) into Eqs. (72) and (73) respectively, we get

$$i(\partial_{\alpha} + iqA_{\alpha})\rho^{\alpha}\,\psi_{\mu} = 0\,, \tag{79}$$

$$i\overline{\psi}^{\mu}\rho^{\alpha}(\overleftarrow{\partial_{\alpha}} - iqA_{\alpha}) = 0\,. \tag{80}$$

Also from Eqs. (67) and (69 - 71), we get the following equations of motion:

$$\dot{p}_{\mu} = \frac{1}{\pi}\,\partial^i\partial_i X^{\mu}\,, \tag{81}$$

$$\dot{p}^{\mu}_{\psi} = -\frac{i}{2\pi}\,\overline{\psi}^{\,\mu}(\overleftarrow{\partial_i}\rho^i + iq\,\rho^{\alpha}\,A_{\alpha})\,, \tag{82}$$

$$\dot{\pi}^i = \frac{1}{2\pi}\,(2\partial^l F_{il} - q\,\overline{\psi}^{\,\mu}\,\rho^i\,\psi_{\mu})\,, \tag{83}$$

$$\dot{\pi}^0 = \partial_i\pi^i - \frac{1}{2\pi}\,q\,\overline{\psi}^{\,\mu}\,\rho^0\,\psi_{\mu}\,. \tag{84}$$

Substituting from Eq.(79) into Eq.(68), we have

$$\dot{p}^{\mu}_{\overline{\psi}} = 0\,. \tag{85}$$

As a comparison between the above two methods, we get that the Hamilton-Jacobi method and Dirac's method give the same equations of motion.

5 Path Integral Quantization of The Relativistic Local Free Field Theory

As an example of a singular system described by a first-order action, namely a system whose Lagrange function is linear in the velocities. However, the associated constraints are all second-class. Let us consider the relativistic local free field theory of spin $\frac{1}{2}$ in a Minkowski spacetime of dimension D. As usual, spacetime coordinates are denoted as $x^{\mu}, y^{\mu} (\mu = 0, 1, \ldots, D-1)$ and space components are labelled by $i, j = 1, 2, \ldots, D-1$. The Minkowski matrix $\eta^{\mu\nu}$ is chosen with a signature with mostly minus signs, and we also set $\hbar = c = 1$. The system is described by the first-order action

$$S[\psi] = \int d^D x\, l(\psi, \partial_{\mu}\psi). \tag{86}$$

with the local lagrangian density function

$$l(\psi, \partial_{\mu}\psi) = i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi + i\frac{\lambda-1}{2}\,\partial_{\mu}\overline{\psi}\gamma^{\mu}\psi - m\overline{\psi}\psi\,. \tag{87}$$

Here λ is a parameter, the matrices γ^{μ} define the Dirac algebra in D-dimensional Minkowski spacetime

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\,\eta^{\mu\nu}, \quad \gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0\,, \tag{88}$$

and $\psi_\alpha(x)(\alpha = 1, 2, \ldots, 2^{[D/2]})$ are Grassmann even degrees of freedom defining a Dirac spinor, with

$$\overline{\psi}(x) = \psi^\dagger(x)\,\gamma^0\,. \tag{89}$$

For simplicity, the fields $\psi(x)$ are assumed to fall off sufficiently rapidly at infinity for all practical purposes.
The Lagrangian (87) is singular, since the rank of the Hess matrix (1) is zero.
Let us first discuss the system using Hamilton-Jacobi approach. In this approach the canonical momenta (2) and (3) take the forms

$$p = \frac{\partial L}{\partial \partial_0 \psi} = i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^0 = -H\,, \tag{90}$$

and

$$\overline{p} = \frac{\partial L}{\partial \partial_0 \overline{\psi}} = i\frac{\lambda-1}{2}\,\gamma^0\psi = -\overline{H}\,. \tag{91}$$

where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual ψ as a column vector and $\overline{\psi}$ as a row vector implies that p will be a row vector while $\overline{p}$ will be a column vector.
The usual Hamiltonian H_0 is given as

$$H_0 = -L + \partial_0\psi\, p_\mu + \partial_0\overline{\psi}\, p_{\overline{\mu}}\,\Bigg|_{p_\mu=-H_\mu,\, p_{\overline{\mu}}=-H_{\overline{\mu}}}\,, \tag{92}$$

or,

$$H_0 = -i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2}\,\partial_a\overline{\psi}\gamma^a\psi + m\overline{\psi}\psi\,, \qquad a = 1, 2, 3\,. \tag{93}$$

The set of Hamilton-Jacobi partial differential equation (HJPDE) are

$$H_0' = p_0 + H_0 = p_0 - i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2}\,\partial_a\overline{\psi}\gamma^a\psi + m\overline{\psi}\psi\,, \tag{94}$$

$$H' = p + H = p - i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^0 = 0\,, \tag{95}$$

$$\overline{H'} = \overline{p} + \overline{H} = \overline{p} - i\frac{\lambda-1}{2}\,\gamma^0\psi = 0\,. \tag{96}$$

Therefor, the total differential equations for the characteristic (14 - 16) are:

$$d\psi = d\psi\,, \tag{97}$$

$$d\overline{\psi} = d\overline{\psi}\,, \tag{98}$$

$$dp = \left(i\frac{\lambda-1}{2}\,\partial_a\overline{\psi}\gamma^a - m\overline{\psi}\right)d\tau + i\frac{\lambda-1}{2}\,\gamma^0 d\overline{\psi}\,, \tag{99}$$

$$d\overline{p} = \left(i\frac{\lambda+1}{2}\,\partial_a\psi\gamma^a - m\psi\right)d\tau + i\frac{\lambda+1}{2}\,\gamma^0 d\psi\,. \tag{100}$$

To check wether the set of equations (97 - 100) is integrable or not, we have to consider the total variation of the constraints. In fact

$$dH' = dp - i\frac{\lambda+1}{2}\,\gamma^0 d\overline{\psi} = 0\,, \tag{101}$$

$$d\overline{H'} = d\overline{p} - i\frac{\lambda-1}{2}\,\gamma^0 d\psi = 0\,. \tag{102}$$

The constraints (95) and (96), lead us to obtain

$$d\overline{\psi} = i(i\partial_a\overline{\psi}\gamma^a + m\overline{\psi})\gamma^0 dt\,, \tag{103}$$

and

$$d\psi = i\gamma^0(i\gamma^a\partial_a\psi - m\psi)dt\,. \tag{104}$$

Then, we conclude that the set of equations (97 - 100) is integrable.

Making use of Eq. (17) and Eqs. (94 - 96), we can write the canonical action integral as

$$Z = \int d^4x\left(i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\lambda-1}{2}\,\partial_\mu\overline{\psi}\gamma^\mu\psi - m\overline{\psi}\psi\right). \tag{105}$$

Now the S-matrix element is given by

$$\left\langle \psi, \overline{\psi}, \iota; \psi', \overline{\psi}', \iota \right\rangle = \int d\psi\, d\overline{\psi}$$
$$exp\left[i\left\{\int d^4x\left(i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\lambda-1}{2}\,\partial_\mu\overline{\psi}\gamma^\mu\psi - m\overline{\psi}\psi\right)\right\}\right]. \tag{106}$$

Now we will apply the Senjanovic method to the previous example. The total Hamiltonian is given as

$$H_T = H_0 + \nu H' + \overline{\nu}\overline{H'}\,, \tag{107}$$

or

$$H_T = -i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2}\,\partial_a\overline{\psi}\gamma^a\psi + m\overline{\psi}\psi \\ + \nu(p - i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^0) + \overline{\nu}(\overline{p} - i\frac{\lambda-1}{2}\,\gamma^0\psi)\,, \tag{108}$$

where ν and $\overline{\nu}$ are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

$$\dot{H'} = \{H', H_T\} = -i\partial_a\overline{\psi}\,\gamma^a - m\overline{\psi} - i\,\overline{\nu}\,\gamma^0 \approx 0\,, \tag{109}$$

$$\dot{\overline{H}'} = \{\overline{H}', H_T\} = i\,\gamma^a\,\partial_a\psi - m\,\psi + i\,\gamma^0\nu \approx 0\,. \tag{110}$$

Eqs. (109) and (110) fix the multipliers $\overline{\nu}$ and ν, respectively as

$$\overline{\nu} = -\overline{\psi}(\overleftarrow{\partial_a}\gamma^a - im)\gamma^0\,, \tag{111}$$

and

$$\nu = -\gamma^0(\gamma^a\,\partial_a + im)\,\psi\,. \tag{112}$$

There are no secondary constraints. By taking suitable linear combinations of constraints, one has to find the maximal number of second class only, there are

$$\Phi_1 = H' = p - i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^0\,, \tag{113}$$

and

$$\Phi_2 = \overline{H}' = \overline{p} - i\frac{\lambda-1}{2}\,\gamma^0\psi\,. \tag{114}$$

The total Hamiltonian is vanishing weakly. It can completely be written in terms of second-class constraints as

$$H_T = -i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^a\partial_a\psi - i\frac{\lambda-1}{2}\,\partial_a\overline{\psi}\gamma^a\psi + m\overline{\psi}\psi + \nu\,\Phi_1 + \overline{\nu}\Phi_2\,. \tag{115}$$

The equations of motion are read as

$$\dot{\psi} = \{\psi, H_T\} = \nu\,, \tag{116}$$

$$\dot{\overline{\psi}} = \{\overline{\psi}, H_T\} = \overline{\nu}\,, \tag{117}$$

$$\dot{p} = \{p, H_T\} = -i\partial_a\overline{\psi}\gamma^a - m\overline{\psi} + i\overline{\nu}\,\frac{\lambda-1}{2}\,\gamma^0\,, \tag{118}$$

and

$$\dot{\overline{p}} = \{\overline{p}, H_T\} = i\gamma^a\partial_a\psi - m\psi + i\nu\,\frac{\lambda+1}{2}\,\gamma^0\,. \tag{119}$$

To obtain the path integral quantization, taking into our consideration that we have two constraints (primary constraint), which are second-class constraints, then we make use the Senjanovic method Eq. (22) one obtains

$$\begin{aligned}\langle Out|S|In\rangle = \int exp\Big[i\int_{-\infty}^{+\infty}\Big(i\frac{\lambda+1}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi + i\frac{\lambda-1}{2}\,\partial_\mu\overline{\psi}\gamma^\mu\psi - m\overline{\psi}\psi\Big)\Big]\\ dt\,D\psi\,DpD\overline{\psi}\;D\overline{p}\;\det(\gamma^0 I)\\ \times\;\delta(p - i\frac{\lambda+1}{2}\,\overline{\psi}\gamma^0)\,\delta(\overline{p} - i\frac{\lambda-1}{2}\gamma^0\psi)\,.\end{aligned} \tag{120}$$

After integrating over p and $\overline{p}$ one can arrive at the result which has seen in Eq. (106).

6 Hamilton-Jacobi quantization of the scalar field coupled to two fLavours of fermions through yukawa couplings

We consider one loop order the self-energy for the scalar field φ with a mass m, coupled to two flavours of fermions with masses m_1 and m_2, coupled through Yukawa couplings described by the lagrangian

$$\begin{aligned}L = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 + \sum_i \overline{\psi}_{(i)}(i\gamma^\mu\partial_\mu - m_i)\psi_{(i)}\\ - g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)}), \qquad \mu = 0,1,2,3\,,\end{aligned} \tag{121}$$

where λ is parameter and g constant, φ, $\psi_{(i)}$, and $\overline{\psi}_{(i)}$ are odd ones. We are adopting the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$.

The Lagrangian function (121) is singular, since the rank of the Hess matrix (1) is one. The generalized momenta (2, 3) are

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \partial^0 \varphi \,, \tag{122}$$

$$p_{(i)} = \frac{\partial L}{\partial \dot{\psi}_{(i)}} = i\overline{\psi}_{(i)}\gamma^0 = -H_{(i)}, \qquad i = 1, 2\,, \tag{123}$$

$$\overline{p}_{(i)} = \frac{\partial L}{\partial \dot{\overline{\psi}}_{(i)}} = 0 = -\overline{H}_{(i)}\,. \tag{124}$$

Where we must call attention to the necessity of being careful with the spinor indexes. Considering, as usual $\psi_{(i)}$ as a column vector and $\overline{\psi}_{(i)}$ as a row vector implies that $p_{(i)}$ will be a row vector while $\overline{p}_{(i)}$ will be a column vector.

Since the rank of the Hess matrix is one, one may solve (122) for $\partial^0\varphi$ as

$$\partial^0 \varphi = p_\varphi \equiv \omega\,. \tag{125}$$

The usual Hamiltonian H_0 is given as

$$H_0 = -L + \omega p_\varphi + \partial_0 \psi_{(i)}\, p_{(i)} \bigg|_{p_{(i)} = -H_{(i)}} + \partial_0 \overline{\psi}_{(i)}\, \overline{p}_{(i)} \bigg|_{\overline{p}_{(i)} = -\overline{H}_{(i)}}, \tag{126}$$

or

$$\begin{aligned} H_0 = \frac{1}{2}({p^2}_\varphi - \partial_a\varphi\partial^a\varphi) + \frac{1}{2}m^2\varphi^2 + \frac{1}{6}\lambda\varphi^3 - \overline{\psi}_{(i)}(i\gamma^a\partial_a - m_i)\psi_{(i)} \\ + g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})\,, \qquad a = 1, 2, 3\,. \end{aligned} \tag{127}$$

The set of Hamilton-Jacobi Partial Differential Equations (7) and (8) read as

$$\begin{aligned} H_0' &= p_0 + H_0 \\ &= p_0 + \frac{1}{2}({p^2}_\varphi - \partial_a\varphi\partial^a\varphi) + \frac{1}{2}m^2\varphi^2 + \frac{1}{6}\lambda\varphi^3 - \overline{\psi}_{(i)}(i\gamma^a\partial_a - m_i)\psi_{(i)} \\ &\quad + g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})\,, \end{aligned} \tag{128}$$

$$H'_{(i)} = p_{(i)} + H_{(i)} = p_{(i)} - i\,\overline{\psi}_{(i)}\,\gamma^0 = 0\,, \tag{129}$$

$$\overline{H}'_{(i)} = \overline{p}_{(i)} + \overline{H}_{(i)} = \overline{p}_{(i)} = 0\,. \tag{130}$$

Therefor, the total differential equations for the characteristic (14 - 16) are:

$$d\varphi = p_\varphi d\tau\,, \tag{131}$$

$$d\psi_{(i)} = d\psi_{(i)}\,, \tag{132}$$

$$d\overline{\psi}_{(i)} = d\overline{\psi}_{(i)}\,, \tag{133}$$

$$dp_\varphi = \left[m^2\varphi + \frac{1}{2}\lambda\varphi^2 + g(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})\right]d\tau\,, \tag{134}$$

$$dp_{(1)} = \left[\overline{\psi}_{(1)}(i\overleftarrow{\partial_a}\gamma^a + m_1) + g\,\varphi\,\overline{\psi}_{(2)}\right]d\tau\,, \tag{135}$$

$$dp_{(2)} = \left[\overline{\psi}_{(2)}(i\overleftarrow{\partial_a}\gamma^a + m_2) + g\,\varphi\,\overline{\psi}_{(1)}\right]d\tau\,, \tag{136}$$

$$d\overline{p}_{(1)} = \left[-(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\varphi\psi_{(2)}\right]d\tau - i\gamma^0 d\psi_{(1)}\,, \tag{137}$$

and

$$d\overline{p}_{(2)} = \left[-(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\varphi\psi_{(1)}\right]d\tau - i\gamma^0 d\psi_{(2)}\,. \tag{138}$$

To check whether the set of equations (131 - 138) is integrable or not, we have to consider the total variations of the constraints. In fact

$$dH'_{(i)} = dp_{(i)} - i\,d\overline{\psi}_{(i)}\,\gamma^0 = 0\,, \tag{139}$$

$$d\overline{H}'_{(i)} = d\overline{p}_{(i)} = 0\,. \tag{140}$$

The constraints (129) and (130), lead us to obtain $d\overline{\psi}_{(i)}$ and $d\psi_{(i)}$ in terms of dt

$$d\overline{\psi}_{(1)} i\gamma^0 = [\overline{\psi}_{(1)}(i\overleftarrow{\partial_a}\gamma^a + m_1) + g\varphi\,\overline{\psi}_{(2)}]dt\,, \tag{141}$$

$$d\overline{\psi}_{(2)} i\gamma^0 = [\overline{\psi}_{(2)}(i\overleftarrow{\partial_a}\gamma^a + m_2) + g\varphi\,\overline{\psi}_{(1)}]dt\,, \tag{142}$$

$$i\gamma^0 d\psi_{(1)} = [-(i\gamma^a\partial_a - m_1)\psi_{(1)} + g\,\varphi\,\psi_{(2)}]d\,, \tag{143}$$

and

$$i\gamma^0 d\psi_{(2)} = [-(i\gamma^a\partial_a - m_2)\psi_{(2)} + g\,\varphi\,\psi_{(1)}]dt\,. \tag{144}$$

We obtain that the set of equations (131 - 138) is integrable. Making use of (17), and (128 - 130), we can write the canonical action integral as

$$Z = \int d^4x[\frac{1}{2}({p^2}_{\varphi} + \partial_a\varphi\partial^a\varphi) - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 + \overline{\psi}_{(i)}(i\gamma^\mu\partial_\mu - m_i)\psi_{(i)} - g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})]\,, \tag{145}$$

Now the path integral representation (24) is given by

$$\langle out|S|In\rangle = \int \prod_i^2 d\varphi\, dp_\varphi\, d\psi_{(i)}\, d\overline{\psi}_{(i)}\; exp\left\{ i\left[\int d^4x \frac{1}{2}({p^2}_{\varphi} + \partial_a\varphi\partial^a\varphi) - \frac{1}{2}m^2\varphi^2 - \frac{1}{6}\lambda\varphi^3 + \overline{\psi}_{(i)}(i\gamma^\mu\partial_\mu - m_i)\psi_{(i)} - g\varphi(\overline{\psi}_{(1)}\psi_{(2)} + \overline{\psi}_{(2)}\psi_{(1)})\right]\right\}. \tag{146}$$

7 Conclusion

In this paper, we have investigated three different constrained systems. Two of them are studied by using Dirac's Hamiltonian formalism and Hamilton-Jacobi approach. The third one quantized by Hamilton-Jacobi quantization.

We have treated constrained system of the Lagrangian describing superstring and have obtained the equations of motion of this system by Dirac's and Hamilton-Jacobi method. In the Dirac method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to drive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way. However, in the Hamilton-Jacobi formalism, there is no need to introduce Lagrange multipliers to the canonical Hamiltonian. Both the consistency conditions and integrability conditions lead to the same constraints. In the Hamilton-Jacobi formulation, the equations of motion are obtained directly by using HJPDES as total differential equations.

Path integral quantization of the relativistic local free field theory is obtained by using the Senjanovic method and the Hamilton-Jacobi path integral formulation. Both methods give the same results. However, in the Hamilton-Jacobi path integral formulation, since the integrability conditions dH' and $d\overline{H}'$ are satisfied, so this system is integrable, and hence the path integral is obtained directly as an integration over the canonical phase-space coordinates $(\psi, \overline{\psi})$. In the usual formulation, one has to integrate over the extended phase-space $(p, \psi, \overline{p}, \overline{\psi})$ and one can get rid of the redundant variables $(p, \overline{p})$ by using delta function $\delta(p - i\frac{\lambda+1}{2}\overline{\psi}\gamma^0)$ and $\delta(\overline{p} - i\frac{\lambda-1}{2}\gamma^0\psi)$. Furthermore, the scalar field coupled to two flavours of fermions through Yukawa couplings are quantized as a constrained system by using Hamilton-Jacobi quantization. That is no need to introduce Lagrange multipliers to the canonical Hamiltonian, then the Hamilton-Jacobi is simpler and more economical.

As a conclusion, the Hamilton-Jacobi approach is always in exact agreement with Dirac's method. Both the consistency conditions and integrability conditions lead to the same constraints. The singular system with second-class constraints is quantized by Hamilton-Jacobi quantization successfully. The Hamilton-Jacobi path integral quantization is simpler and more economical. In Hamilton-Jacobi treatment, there is no need to distinguish between first-class and second-class constraints, and there is no need to introduce Lagrange multipliers; all that is needed is the set of Hamilton-Jacobi partial differential equations and the equations of motion. If the system is integrable then one can construct the canonical phase space. In hamilton-Jacobi quantization, the gauge fixing is not necessary to obtain the path integral formulation for field theories if the canonical formulation is used. Since this system is integrable, the path integral is obtained as an integration over the canonical phase-space coordinates.

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