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MAPPINGS PRESERVING SUM OF PRODUCTS ab – b o a* ON FACTOR VON NEUMANN ALGEBRAS

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Abstract

Let $\mathscr{A}$ and $\mathscr{B}$ be two factor von Neumann algebras. In this paper, we proved that a bijective mapping $\Phi : \mathscr{A} \to \mathscr{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$ (where $\circ$ is the special Jordan product on $\mathscr{A}$ and $\mathscr{B}$, respectively), for all elements $a, b \in \mathscr{A}$, if and only if Φ is a $*$-ring isomorphism. In particular, if the von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism.

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1 Introduction

Let $\mathscr{R}$ and $\mathscr{S}$ be rings. We say that a mapping $\Phi : \mathscr{R} \to \mathscr{S}$ *preserves product* if $\Phi(ab) = \Phi(a)\Phi(b)$, for all elements $a, b \in \mathscr{R}$, *preserves Jordan product* if $\Phi(ab + ba) = \Phi(a)\Phi(b) + \Phi(b)\Phi(a)$, for all elements $a, b \in \mathscr{R}$, and *preserves Lie product* if $\Phi(ab - ba) = \Phi(a)\Phi(b) - \Phi(b)\Phi(a)$, for all elements $a, b \in \mathscr{R}$. We say that a mapping $\Phi : \mathscr{R} \to \mathscr{S}$ is *additive* if $\Phi(a + b) = \Phi(a) + \Phi(b)$, for all elements $a, b \in \mathscr{R}$ and that is a *ring isomorphism* if Φ is an additive bijection that preserves products.

Let $\mathscr{R}$ and $\mathscr{S}$ be $*$-rings. We say that a mapping $\Phi : \mathscr{R} \to \mathscr{S}$ *preserves involution* if $\Phi(a^*) = \Phi(a)^*$, for all elements $a \in \mathscr{R}$, and that Φ is a $*$*-ring isomorphism* if Φ is a ring isomorphism that preserves involution.

In recent years, there has been a great interest in the study of additivity of mappings. A part of these results is focused on the additivity of mappings that preserve products, Jordan products and Lie products on rings as well as operator algebras (for example, see [2], [5], [6] and [7]).

As in the case of rings, a natural problem on $*$-rings is to know when mappings defined on them preserving some new type of product are $*$-ring isomorphisms. Recently, many mathematicians devoted themselves to study mappings preserving new products on rings with involution (for example, see [1], [3] and [4]). In particular, Cui and Li [1] studied the bijective mappings preserving the new product $ab - ba^*$. They showed that such mappings on factor von Neumann algebras are $*$-ring isomorphisms.

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively, and $\mathscr{A}$ and $\mathscr{B}$ be two algebras over $\mathbb{C}$. If $a, b \in \mathscr{A}$, let $a \circ b = \frac{1}{2}(ab + ba)$. We call $\circ$ the *special Jordan product* on $\mathscr{A}$. We say that a mapping $\Phi : \mathscr{A} \to \mathscr{B}$ preserves *special Jordan product* if $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$, for all elements $a, b \in \mathscr{A}$.

A *von Neumann algebra* $\mathscr{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathscr{H}$, which are denote in this paper by lowercase Latin letters, containing the identity operator $1_{\mathscr{A}}$, called *elements of* $\mathscr{A}$. A von Neumann algebra $\mathscr{A}$ is called *factor* if its center contains the multiples of the identity operator only. It is well known that the factor von Neumann algebra $\mathscr{A}$ is prime, that is, if a and b are elements of $\mathscr{A}$ such that $a\mathscr{A}b = 0$, then either $a = 0$ or $b = 0$.

Let $\mathscr{A}$ and $\mathscr{B}$ be two $*$-algebras. We say that a mapping $\Phi : \mathscr{A} \to \mathscr{B}$ *preserves sum of products* $ab - b \circ a^*$ if

$$\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*, \tag{1}$$

for all elements $a, b \in \mathscr{A}$.

Similarly to the research performed in [1], the aim of the present paper is to investigate when a bijective mapping preserving sum of products $ab - b \circ a^*$ on factor von Neumann algebras is a $*$-ring isomorphism. Our main result reads as follows.

Main Theorem. *Let $\mathscr{A}$ and $\mathscr{B}$ be two factor von Neumann algebras with $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$ the identities of them, respectively. Then every bijective mapping $\Phi : \mathscr{A} \to \mathscr{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$, for all elements $a, b \in \mathscr{A}$, if and only if Φ is a $*$-ring isomorphism.*

2 The proof of the main theorem

The proof of the Main Theorem is made by proving several lemmas. We begin with the following lemma.

Lemma 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two von Neumann algebras and $\Phi : \mathscr{A} \to \mathscr{B}$ a mapping that preserve sum of products $ab - b \circ a^*$. If $\Phi(c) = \Phi(a) + \Phi(b)$, for elements a, b, c of $\mathscr{A}$, then hold the following identities: (i) $\Phi(tc - c \circ t^*) = \Phi(ta - a \circ t^*) + \Phi(tb - b \circ t^*)$ and (ii) $\Phi(ct - t \circ c^*) = \Phi(at - t \circ a^*) + \Phi(bt - t \circ b^*)$, for all elements t of $\mathscr{A}$.*

Proof. For any element t of $\mathscr{A}$, we have

$$\begin{aligned} \Phi(tc - c \circ t^*) &= \Phi(t)\Phi(c) - \Phi(c) \circ \Phi(t)^* \\ &= \Phi(t)(\Phi(a) + \Phi(b)) - (\Phi(a) + \Phi(b)) \circ \Phi(t)^* \\ &= \Phi(t)\Phi(a) - \Phi(a) \circ \Phi(t)^* + \Phi(t)\Phi(b) - \Phi(b) \circ \Phi(t)^* \\ &= \Phi(ta - a \circ t^*) + \Phi(tb - b \circ t^*). \end{aligned}$$

Correspondingly, we prove (ii). □

Lemma 2.2. $\Phi(0) = 0$.

Proof. From supposition of Φ, there is an element $b \in \mathscr{A}$ such that $\Phi(b) = 0$. This results that

$$\Phi(0) = \Phi(0b - b \circ 0^*) = \Phi(0)\Phi(b) - \Phi(b) \circ \Phi(0)^* = \Phi(0)0 - 0 \circ \Phi(0)^* = 0.$$

□

Lemma 2.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two factor von Neumann algebras with $1_{\mathscr{A}}$ the identity of $\mathscr{A}$. Then every bijective mapping $\Phi : \mathscr{A} \to \mathscr{B}$ that preserve sum of products $ab - b \circ a^*$ is additive.*

We will establish the proof of Lemma 2.3 in a series of claims based on the techniques used by Cui and Li [1]. We begin, though, with a well-known result that will be used throughout this paper.

Let p_1 be an arbitrary non-trivial projection of $\mathscr{A}$ and write $p_2 = 1 - p_1$. Then $\mathscr{A}$ has a Peirce decomposition $\mathscr{A} = \mathscr{A}_{11} \oplus \mathscr{A}_{12} \oplus \mathscr{A}_{21} \oplus \mathscr{A}_{22}$, where $\mathscr{A}_{ij} = p_i \mathscr{A} p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathscr{A}_{ij}\mathscr{A}_{kl} \subseteq \delta_{jk}\mathscr{A}_{il}$, where δ_{jk} is the *Kronecker delta function.*

Claim 2.1. *For any elements $a_{11} \in \mathscr{A}_{11}$, $a_{22} \in \mathscr{A}_{22}$, $b_{12} \in \mathscr{A}_{12}$ and $b_{21} \in \mathscr{A}_{21}$ hold: (i) $\Phi(a_{11} + b_{12}) = \Phi(a_{11}) + \Phi(b_{12})$, (ii) $\Phi(a_{11} + b_{21}) = \Phi(a_{11}) + \Phi(b_{21})$, (iii) $\Phi(a_{22} + b_{12}) = \Phi(a_{22}) + \Phi(b_{12})$ and (iv) $\Phi(a_{22} + b_{21}) = \Phi(a_{22}) + \Phi(b_{21})$.*

Proof. From supposition of Φ, consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathscr{A}$ verifying $\Phi(c) = \Phi(a_{11}) + \Phi(b_{12})$. By Lemma 2.1(i) we have

$$\begin{aligned}\Phi(p_2 c - c \circ p_2^*) &= \Phi(p_2 a_{11} - a_{11} \circ p_2^*) + \Phi(p_2 b_{12} - b_{12} \circ p_2^*) \\ &= \Phi(p_2 b_{12} - b_{12} \circ p_2^*).\end{aligned}$$

On account of this we have that $p_2 c - c \circ p_2^* = -\frac{1}{2} b_{12}$ which implies that $c_{12} = b_{12}$ and $c_{21} = 0$. This shows that $\Phi(c_{11} + b_{12} + c_{22}) = \Phi(a_{11}) + \Phi(b_{12})$. Hence, for any element $d_{12} \in \mathscr{A}_{12}$ we have

$$\begin{aligned}\Phi(d_{12}(c_{11} + b_{12} + c_{22}) - (c_{11} + b_{12} + c_{22}) \circ d_{12}^*) &= \Phi(d_{12} a_{11} - a_{11} \circ d_{12}^*) \\ &\quad + \Phi(d_{12} b_{12} - b_{12} \circ d_{12}^*)\end{aligned}$$

which results that

$$\Phi(d_{12}c_{22} - \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) = \Phi(-\frac{1}{2}d_{12}^*a_{11}) + \Phi(-b_{12} \circ d_{12}^*).$$

It follows that

$$\begin{aligned}\Phi(p_2(d_{12}c_{22} &- \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) \\ &- (d_{12}c_{22} - \frac{1}{2}(b_{12}d_{12}^* + c_{22}d_{12}^* + d_{12}^*c_{11} + d_{12}^*b_{12})) \circ p_2^*) \\ = \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) &- (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*) + \Phi(p_2(-b_{12} \circ d_{12}^*) - (-b_{12} \circ d_{12}^*) \circ p_2^*)\end{aligned}$$

which leads to

$$\Phi(-\frac{1}{2}d_{12}c_{22} - \frac{1}{2^2}(c_{22}d_{12}^* + d_{12}^*c_{11})) = \Phi(-\frac{1}{2^2}d_{12}^*a_{11}).$$

This implies that $-\frac{1}{2}d_{12}c_{22} - \frac{1}{2^2}(c_{22}d_{12}^* + d_{12}^*c_{11}) = -\frac{1}{2^2}d_{12}^*a_{11}$ which results in $c_{11} = a_{11}$ and $c_{22} = 0$.

Likewise, we prove the cases (ii), (iii) and (iv). □

Claim 2.2. *For any elements $a_{12} \in \mathscr{A}_{12}$ and $b_{21} \in \mathscr{A}_{21}$ holds $\Phi(a_{12} + b_{21}) = \Phi(a_{12}) + \Phi(b_{21})$.*

Proof. From supposition of Φ, consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathscr{A}$ satisfying $\Phi(c) = \Phi(a_{12}) + \Phi(b_{21})$. For any element $d_{12} \in \mathscr{A}_{12}$, we have

$$\begin{aligned}\Phi(d_{12}c - c \circ d_{12}^*) &= \Phi(d_{12}a_{12} - a_{12} \circ d_{12}^*) + \Phi(d_{12}b_{21} - b_{21} \circ d_{12}^*) \\ &= \Phi(-(a_{12} \circ d_{12}^*)) + \Phi(d_{12}b_{21})\end{aligned}$$

which implies that

$$\begin{aligned}\Phi(p_2(d_{12}c &- c \circ d_{12}^*) - (d_{12}c - c \circ d_{12}^*) \circ p_2^*) \\ &= \Phi(p_2(-(a_{12} \circ d_{12}^*)) - (-(a_{12} \circ d_{12}^*)) \circ p_2^*)\end{aligned}$$

$$+\Phi(p_2(d_{12}b_{21})-(d_{12}b_{21})\circ p_2^*)=0.$$

This leads to $p_2(d_{12}c-c\circ d_{12}^*)-(d_{12}c-c\circ d_{12}^*)\circ p_2^*=0$ that produces $c_{11}=c_{22}=0$. As result, we obtain

$$\Phi(c_{12}+c_{21})=\Phi(a_{12})+\Phi(b_{21}).$$

It follows that, for any element $d_{12}\in\mathscr{A}_{12}$ we have

$$\Phi(d_{12}(c_{12}+c_{21})-(c_{12}+c_{21})\circ d_{12}^*)=\Phi(d_{12}a_{12}-a_{12}\circ d_{12}^*)$$
$$+\Phi(d_{12}b_{21}-b_{21}\circ d_{12}^*)=\Phi(-(a_{12}\circ d_{12}^*))+\Phi(d_{12}b_{21}).$$

On account of this, we obtain

$$\Phi((ip_2)(d_{12}c_{21}-c_{12}\circ d_{12}^*)-(d_{12}c_{21}-c_{12}\circ d_{12}^*)\circ(ip_2)^*)$$
$$=\Phi((ip_2)(-(a_{12}\circ d_{12}^*))-(-(a_{12}\circ d_{12}^*))\circ(ip_2)^*)+\Phi((ip_2)(d_{12}b_{21})$$
$$-(d_{12}b_{21})\circ(ip_2)^*)=\Phi((ip_2)(-(a_{12}\circ d_{12}^*))-(-(a_{12}\circ d_{12}^*))\circ(ip_2)^*).$$

which implies that

$$(ip_2)(d_{12}c_{21}-c_{12}\circ d_{12}^*)-(d_{12}c_{21}-c_{12}\circ d_{12}^*)\circ(ip_2)^*$$
$$=(ip_2)(-(a_{12}\circ d_{12}^*))-(-(a_{12}\circ d_{12}^*))\circ(ip_2)^*$$

and resulting in $d_{12}^*c_{12}=d_{12}^*a_{12}$. It shows that $c_{12}=a_{12}$. Next, for any element $d_{21}\in\mathscr{A}_{21}$

$$\Phi(d_{21}(a_{12}+c_{21})-(a_{12}+c_{21})\circ d_{21}^*)=\Phi(d_{21}a_{12}-a_{12}\circ d_{21}^*)$$
$$+\Phi(d_{21}b_{21}-b_{21}\circ d_{21}^*)=\Phi(d_{21}a_{12})+\Phi(-(b_{21}\circ d_{21}^*)).$$

As a results, we obtain

$$\Phi((ip_1)(d_{21}a_{12}-c_{21}\circ d_{21}^*)-(d_{21}a_{12}-c_{21}\circ d_{21}^*)\circ(ip_1)^*)$$
$$=\Phi((ip_1)(d_{21}a_{12})-(d_{21}a_{12})\circ(ip_1)^*)+\Phi((ip_1)(-(b_{21}\circ d_{21}^*))$$
$$-(-(b_{21}\circ d_{21}^*))\circ(ip_1)^*)=\Phi((ip_1)(-(b_{21}\circ d_{21}^*))-(-(b_{21}\circ d_{21}^*))\circ(ip_1)^*)$$

which leads to

$$(ip_1)(d_{21}a_{12}-c_{21}\circ d_{21}^*)-(d_{21}a_{12}-c_{21}\circ d_{21}^*)\circ(ip_1)^*$$
$$=(ip_1)(-(b_{21}\circ d_{21}^*))-(-(b_{21}\circ d_{21}^*))\circ(ip_1)^*.$$

This implies that $d_{21}^*c_{21}=d_{21}^*b_{21}$ which results in $c_{21}=b_{21}$. $\square$

Claim 2.3. *For any elements $a_{11} \in \mathscr{A}_{11}$, $b_{12} \in \mathscr{A}_{12}$, $c_{21} \in \mathscr{A}_{21}$ and $d_{22} \in \mathscr{A}_{22}$ hold: (i) $\Phi(a_{11}+b_{12}+c_{21}) = \Phi(a_{11})+\Phi(b_{12})+\Phi(c_{21})$ and (ii) $\Phi(b_{12}+c_{21}+d_{22}) = \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.*

Proof. Consider an element $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathscr{A}$ verifying $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21})$. By Claim 2.2 we have

$$\begin{aligned}\Phi(p_2 f - f \circ p_2^*) &= \Phi(p_2 a_{11} - a_{11} \circ p_2^*) + \Phi(p_2 b_{12} - b_{12} \circ p_2^*) + \Phi(p_2 c_{21} - c_{21} \circ p_2^*) \\ &= \Phi(-\frac{1}{2} b_{12}) + \Phi(\frac{1}{2} c_{21}) = \Phi(-\frac{1}{2} b_{12} + \frac{1}{2} c_{21})\end{aligned}$$

which implies that $p_2 f - f \circ p_2^* = -\frac{1}{2} b_{12} + \frac{1}{2} c_{21}$. As result we obtain $f_{12} = b_{12}$ and $f_{21} = c_{21}$. Hence,

$$\Phi(f_{11} + b_{12} + c_{21} + f_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}).$$

As a consequence, we have

$$\begin{aligned}&\Phi((ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^*) \\ &= \Phi((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \Phi((ip_2)b_{12} - b_{12} \circ (ip_2)^*) + \Phi((ip_2)c_{21} - c_{21} \circ (ip_2)^*)\end{aligned}$$

which results that

$$\begin{aligned}&\Phi((ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^*) \\ &\qquad = \Phi(\frac{1}{2} i b_{12}) + \Phi(\frac{3}{2} i c_{21}) = \Phi(i(\frac{1}{2} b_{12} + \frac{3}{2} c_{21})).\end{aligned}$$

This shows that $(ip_2)(f_{11} + b_{12} + c_{21} + f_{22}) - (f_{11} + b_{12} + c_{21} + f_{22}) \circ (ip_2)^* = i(\frac{1}{2} b_{12} + \frac{3}{2} c_{21})$ which yields $f_{22} = 0$. It follows that

$$\begin{aligned}&\Phi(d_{12}(f_{11} + b_{12} + c_{21}) - (f_{11} + b_{12} + c_{21}) \circ d_{12}^*) \\ &\quad = \Phi(d_{12} a_{11} - a_{11} \circ d_{12}^*) + \Phi(d_{12} b_{12} - b_{12} \circ d_{12}^*) + \Phi(d_{12} c_{21} - c_{21} \circ d_{12}^*) \\ &\qquad = \Phi(-\frac{1}{2} d_{12}^* a_{11}) + \Phi(-(b_{12} \circ d_{12}^*)) + \Phi(d_{12} c_{21})\end{aligned}$$

which implies that

$$\Phi(p_2(d_{12}c_{21} - \frac{1}{2}d_{12}^*f_{11} - b_{12} \circ d_{12}^*) - (d_{12}c_{21} - \frac{1}{2}d_{12}^*f_{11} - b_{12} \circ d_{12}^*) \circ p_2^*)$$
$$= \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) - (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*) + \Phi(p_2(-(b_{12} \circ d_{12}^*)) - (-(b_{12} \circ d_{12}^*)) \circ p_2^*) + \Phi(p_2(d_{12}c_{21}) - (d_{12}c_{21}) \circ p_2^*)$$
$$= \Phi(p_2(-\frac{1}{2}d_{12}^*a_{11}) - (-\frac{1}{2}d_{12}^*a_{11}) \circ p_2^*).$$

On account of this, we obtain $d_{12}^*f_{11} = d_{12}^*a_{11}$ which implies that $f_{11} = a_{11}$.
Likewise, we prove the case (ii). □

Claim 2.4. *For any elements $a_{11} \in \mathscr{A}_{11}$, $b_{12} \in \mathscr{A}_{12}$, $c_{21} \in \mathscr{A}_{21}$ and $d_{22} \in \mathscr{A}_{22}$ holds $\Phi(a_{11} + b_{12} + c_{21} + d_{22}) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$.*

Proof. Consider an element $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathscr{A}$ verifying $\Phi(f) = \Phi(a_{11}) + \Phi(b_{12}) + \Phi(c_{21}) + \Phi(d_{22})$. By Claims 2.1(i) and 2.3(i) we have

$$\Phi((ip_1)f - f \circ (ip_1)^*) = \Phi((ip_1)a_{11} - a_{11} \circ (ip_1)^*) + \Phi((ip_1)b_{12} - b_{12} \circ (ip_1)^*) + \Phi((ip_1)c_{21} - c_{21} \circ (ip_1)^*) + \Phi((ip_1)d_{22} - d_{22} \circ (ip_1)^*)$$
$$= \Phi(2ia_{11}) + \Phi(\frac{3}{2}ib_{12}) + \Phi(\frac{1}{2}ic_{21}) = \Phi(i(2a_{11} + \frac{3}{2}b_{12} + \frac{1}{2}c_{21})).$$

It follows that $(ip_1)f - f \circ (ip_1)^* = i(2a_{11} + \frac{3}{2}b_{12} + \frac{1}{2}c_{21})$ which implies that $f_{11} = a_{11}$, $f_{12} = b_{12}$ and $f_{21} = c_{21}$. Also,

$$\Phi((ip_2)f - f \circ (ip_2)^*) = \Phi((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \Phi((ip_2)b_{12} - b_{12} \circ (ip_2)^*) + \Phi((ip_2)c_{21} - c_{21} \circ (ip_2)^*) + \Phi((ip_2)d_{22} - d_{22} \circ (ip_2)^*)$$
$$= \Phi(\frac{1}{2}ib_{12}) + \Phi(\frac{3}{2}ic_{21}) + \Phi(2id_{22}) = \Phi(i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21} + 2d_{22})).$$

This results that $(ip_2)f - f \circ (ip_2)^* = i(\frac{1}{2}b_{12} + \frac{3}{2}c_{21} + 2d_{22})$ which implies that $f_{22} = d_{22}$. □

Claim 2.5. *For any elements $a_{12}, b_{12} \in \mathscr{A}_{12}$ and $a_{21}, b_{21} \in \mathscr{A}_{21}$ hold: (i) $\Phi(a_{12} + b_{12}) = \Phi(a_{12}) + \Phi(b_{12})$ and (ii) $\Phi(a_{21} + b_{21}) = \Phi(a_{21}) + \Phi(b_{21})$.*

Proof. First, we note that the following identity is valid

$$(p_1 + a_{12})(p_2 + b_{12}) - (p_2 + b_{12}) \circ (p_1 + a_{12})^* = a_{12} + \frac{1}{2}b_{12} - \frac{1}{2}a_{12}^* - \frac{1}{2}b_{12}a_{12}^* - \frac{1}{2}a_{12}^*b_{12}.$$

On account of this, by Claim 2.4 we have

$$\begin{aligned}
& \Phi(a_{12} + \tfrac{1}{2}b_{12}) + \Phi(-\tfrac{1}{2}a_{12}^*) + \Phi(-\tfrac{1}{2}b_{12}a_{12}^*) + \Phi(-\tfrac{1}{2}a_{12}^*b_{12}) \\
= \; & \Phi(a_{12} + \tfrac{1}{2}b_{12} - \tfrac{1}{2}a_{12}^* - \tfrac{1}{2}b_{12}a_{12}^* - \tfrac{1}{2}a_{12}^*b_{12}) \\
= \; & \Phi((p_1 + a_{12})(p_2 + b_{12}) - (p_2 + b_{12}) \circ (p_1 + a_{12})^*) \\
= \; & \Phi(p_1 + a_{12})\Phi(p_2 + b_{12}) - \Phi(p_2 + b_{12}) \circ \Phi(p_1 + a_{12})^* \\
= \; & (\Phi(p_1) + \Phi(a_{12}))(\Phi(p_2) + \Phi(b_{12})) \\
& -(\Phi(p_2) + \Phi(b_{12})) \circ (\Phi(p_1)^* + \Phi(a_{12})^*) \\
= \; & \Phi(p_1)\Phi(p_2) - \Phi(p_2) \circ \Phi(p_1)^* + \Phi(p_1)\Phi(b_{12}) - \Phi(b_{12}) \circ \Phi(p_1)^* \\
& +\Phi(a_{12})\Phi(p_2) - \Phi(p_2) \circ \Phi(a_{12})^* + \Phi(a_{12})\Phi(b_{12}) - \Phi(b_{12}) \circ \Phi(a_{12})^* \\
= \; & \Phi(p_1p_2 - p_2 \circ p_1^*) + \Phi(p_1b_{12} - b_{12} \circ p_1^*) + \Phi(a_{12}p_2 - p_2 \circ a_{12}^*) \\
& +\Phi(a_{12}b_{12} - b_{12} \circ a_{12}^*) \\
= \; & \Phi(\tfrac{1}{2}b_{12}) + \Phi(a_{12} - \tfrac{1}{2}a_{12}^*) + \Phi(-\tfrac{1}{2}b_{12}a_{12}^* - \tfrac{1}{2}a_{12}^*b_{12}) \\
= \; & \Phi(a_{12}) + \Phi(\tfrac{1}{2}b_{12}) + \Phi(-\tfrac{1}{2}a_{12}^*) + \Phi(-\tfrac{1}{2}b_{12}a_{12}^*) + \Phi(-\tfrac{1}{2}a_{12}^*b_{12}).
\end{aligned}$$

This allows us to conclude that $\Phi(a_{12} + b_{12}) = \Phi(a_{12}) + \Phi(b_{12})$.

Likewise, we prove the case (ii) using the identity

$$(p_2 + a_{21})(p_1 + b_{21}) - (p_1 + b_{21}) \circ (p_2 + a_{21})^* = a_{21} + \frac{1}{2}b_{21} - \frac{1}{2}a_{21}^* - \frac{1}{2}b_{21}a_{21}^* - \frac{1}{2}a_{21}^*b_{21}.$$

□

Claim 2.6. *For any elements $a_{11}, b_{11} \in \mathscr{A}_{11}$ and $a_{22}, b_{22} \in \mathscr{A}_{22}$ hold: (i) $\Phi(a_{11} + b_{11}) = \Phi(a_{11}) + \Phi(b_{11})$ and (ii) $\Phi(a_{22} + b_{22}) = \Phi(a_{22}) + \Phi(b_{22})$.*

Proof. Consider an element $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathscr{A}$ verifying $\Phi(c) = \Phi(a_{11}) + \Phi(b_{11})$. By Claim 2.5(ii), for any element $d_{12} \in \mathscr{A}_{12}$ we have

$$\Phi(d_{12}c - c \circ d_{12}^*) = \Phi(d_{12}a_{11} - a_{11} \circ d_{12}^*) + \Phi(d_{12}b_{11} - b_{11} \circ d_{12}^*)$$
$$= \Phi(-\frac{1}{2}d_{12}^*a_{11}) + \Phi(-\frac{1}{2}d_{12}^*b_{11}) = \Phi(-\frac{1}{2}d_{12}^*(a_{11} + b_{11})).$$

This results that $d_{12}c - c \circ d_{12}^* = -\frac{1}{2}d_{12}^*(a_{11}+b_{11})$ which leads to $c_{11} = a_{11}+b_{11}$ and $c_{12} = c_{21} = c_{22} = 0$.

Likewise, we prove the case (ii). □

Claim 2.7. *Φ is an additive mapping.*

Proof. The result is an immediate consequence of Claims 2.4, 2.5 and 2.6. □

Lemma 2.4. *Let $\mathscr{A}$ be a factor von Neumann algebra with identity $1_{\mathscr{A}}$ and an element $a \in \mathscr{A}$. If $\mathscr{A}$ satisfies the condition: $ab - b \circ a^* = 0$, for all elements $b \in \mathscr{A}$, then $a \in \mathbb{R}1_{\mathscr{A}}$.*

Proof. Note that, when we take $b = 1$, the condition reduces to $a - a^* = 0$. As a consequence, we obtain $ab - ba = 0$, for all elements $b \in \mathscr{A}$. This means that a belongs to the center of $\mathscr{A}$ which leads to $a \in \mathbb{R}1_{\mathscr{A}}$. □

In the remainder of this paper, all lemmas satisfy the conditions of the Main Theorem.

Lemma 2.5. *$\Phi(\mathbb{R}1_{\mathscr{A}}) = \mathbb{R}1_{\mathscr{B}}$, the mapping Φ preserves Hermitian elements in both directions and $\Phi(\mathbb{C}1_{\mathscr{A}}) = \mathbb{C}1_{\mathscr{B}}$.*

Proof. First, observe that for any elements $a, b \in \mathscr{A}$ holds $ab - b \circ a^* = 0$ if and only if $\Phi(a)\Phi(b) - \Phi(a) \circ \Phi(b)^* = 0$. Hence, for any elements $\alpha \in \mathbb{R}$ and $b \in \mathscr{A}$, $0 = \Phi((\alpha 1_{\mathscr{A}})b - b \circ (\alpha 1_{\mathscr{A}})^*) = \Phi(\alpha 1_{\mathscr{A}})\Phi(b) - \Phi(b) \circ \Phi(\alpha 1_{\mathscr{A}})^* = 0$ which implies that $\Phi(\alpha 1_{\mathscr{A}}) \in \mathbb{R}1_{\mathscr{B}}$, by Lemma 2.4. As α is an arbitrary element, we conclude that $\Phi(\mathbb{R}1_{\mathscr{A}}) \subseteq \mathbb{R}1_{\mathscr{B}}$. In view of the fact that Φ^{-1} has the same properties of Φ, then by a similar argument we prove that $\mathbb{R}1_{\mathscr{B}} \subseteq \Phi(\mathbb{R}1_{\mathscr{A}})$. As a consequence, we obtain $\Phi(\mathbb{R}1_{\mathscr{A}}) = \mathbb{R}1_{\mathscr{B}}$.

Let a be an element of $\mathscr{A}$ satisfying the condition $a = a^*$. Then $0 = \Phi(a1_{\mathscr{A}} - 1_{\mathscr{A}} \circ a^*) = \Phi(a)\Phi(1_{\mathscr{A}}) - \Phi(1_{\mathscr{A}}) \circ \Phi(a)^*$ which implies that $\Phi(a) = \Phi(a)^*$. In view of the fact that Φ^{-1} has the same properties of Φ, then by a similar argument we prove that if $\Phi(a) = \Phi(a)^*$, then $a = a^*$, for all such elements $a \in \mathscr{A}$. Therefore, $a = a^*$ if and only if $\Phi(a) = \Phi(a)^*$, for all elements $a \in \mathscr{A}$.

For any elements $\lambda \in \mathbb{C}$ and $a \in \mathscr{A}$ verifying the condition $a = a^*$, we have $0 = \Phi(a(\lambda 1_{\mathscr{A}}) - (\lambda 1_{\mathscr{A}}) \circ a^*) = \Phi(a)\Phi(\lambda 1_{\mathscr{A}}) - \Phi(\lambda 1_{\mathscr{A}}) \circ \Phi(a)^*$ which implies that $\Phi(a)\Phi(\lambda 1_{\mathscr{A}}) - \Phi(\lambda 1_{\mathscr{A}})\Phi(a) = 0$. This results that $b\Phi(\lambda 1_{\mathscr{A}}) - \Phi(\lambda 1_{\mathscr{A}})b = 0$, for any element $b \in \mathscr{B}$ verifying $b = b^*$. As a result, we obtain $b\Phi(\lambda 1_{\mathscr{A}}) - \Phi(\lambda 1_{\mathscr{A}})b = 0$, for any element $b \in \mathscr{B}$, since any element $b \in \mathscr{B}$ can be written as $b = b_1 + ib_2$, where $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2i}$. This shows that $\Phi(\lambda 1_{\mathscr{A}}) \in \mathbb{C}1_{\mathscr{B}}$. As λ is any element, we conclude that $\Phi(\mathbb{C}1_{\mathscr{A}}) \subseteq \mathbb{C}1_{\mathscr{B}}$. Applying a similar argument to mapping Φ^{-1} we obtain $\mathbb{C}1_{\mathscr{B}} \subseteq \Phi(\mathbb{C}1_{\mathscr{A}})$. This means that $\Phi(\mathbb{C}1_{\mathscr{A}}) = \mathbb{C}1_{\mathscr{B}}$. □

Lemma 2.6. *Either $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathscr{A}$, or $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathscr{A}$.*

Proof. From supposition of Φ and by Lemma 2.5, we have $\Phi(\pm\frac{1}{2}i1_{\mathscr{A}}) \in (\mathbb{C} \setminus \mathbb{R})1_{\mathscr{B}}$ and $\Phi(\pm\frac{1}{2}1_{\mathscr{A}}) \in \mathbb{R}1_{\mathscr{B}}$. As $\pm\frac{1}{2}i1_{\mathscr{A}} = (\pm\frac{1}{2}i1_{\mathscr{A}})(\frac{1}{2}1_{\mathscr{A}}) - (\frac{1}{2}1_{\mathscr{A}}) \circ (\pm\frac{1}{2}i1_{\mathscr{A}})^*$, then

$$\begin{aligned}\Phi(\pm\frac{1}{2}i1_{\mathscr{A}}) &= \Phi((\pm\frac{1}{2}i1_{\mathscr{A}})(\frac{1}{2}1_{\mathscr{A}}) - (\frac{1}{2}1_{\mathscr{A}}) \circ (\pm\frac{1}{2}i1_{\mathscr{A}})^*) \\ &= \Phi(\pm\frac{1}{2}i1_{\mathscr{A}})\Phi(\frac{1}{2}1_{\mathscr{A}}) - \Phi(\frac{1}{2}1_{\mathscr{A}}) \circ \Phi(\pm\frac{1}{2}i1_{\mathscr{A}})^*\end{aligned}$$

which implies that $\Phi(\pm\frac{1}{2}i1_{\mathscr{A}})^* = (1_{\mathscr{A}} - \Phi(\frac{1}{2}1_{\mathscr{A}})^{-1})\Phi(\pm\frac{1}{2}i1_{\mathscr{A}})$. This results that $\Phi(\pm\frac{1}{2}i1_{\mathscr{A}}) \in \mathbb{R}i1_{\mathscr{B}}$. As consequence we have

$$\begin{aligned}\Phi(\frac{1}{2}i1_{\mathscr{A}}) &= \Phi((-\frac{1}{2}i1_{\mathscr{A}})(-\frac{1}{2}1_{\mathscr{A}}) - (-\frac{1}{2}1_{\mathscr{A}}) \circ (-\frac{1}{2}i1_{\mathscr{A}})^*) \\ &= \Phi(-\frac{1}{2}i1_{\mathscr{A}})\Phi(-\frac{1}{2}1_{\mathscr{A}}) - \Phi(-\frac{1}{2}1_{\mathscr{A}}) \circ \Phi(-\frac{1}{2}i1_{\mathscr{A}})^* \\ &= 2\Phi(-\frac{1}{2}1_{\mathscr{A}})\Phi(-\frac{1}{2}i1_{\mathscr{A}}), \quad (2)\end{aligned}$$

since $\Phi(\pm\frac{1}{2}i1_{\mathscr{A}})^* = -\Phi(\pm\frac{1}{2}i1_{\mathscr{A}})$, and

$$\Phi(-\frac{1}{2}1_{\mathscr{A}}) = \Phi((-\frac{1}{2}i1_{\mathscr{A}})(-\frac{1}{2}i1_{\mathscr{A}}) - (-\frac{1}{2}i1_{\mathscr{A}}\circ)(-\frac{1}{2}i1_{\mathscr{A}})^*)$$

$$= \Phi(-\frac{1}{2}i1_{\mathscr{A}})\Phi(-\frac{1}{2}i1_{\mathscr{A}}) - \Phi(-\frac{1}{2}i1_{\mathscr{A}}) \circ \Phi(-\frac{1}{2}i1_{\mathscr{A}})^* = 2\Phi(-\frac{1}{2}i1_{\mathscr{A}})^2. \quad (3)$$

By Claim 2.7 and (2)-(3) we conclude that $\Phi(-\frac{1}{2}1_{\mathscr{A}}) = -\frac{1}{2}1_{\mathscr{A}}$ and $\Phi(\frac{1}{2}i1_{\mathscr{A}}) = \pm\frac{1}{2}i1_{\mathscr{A}}$. Thus, for any element $a \in \mathscr{A}$, we have that $\Phi(ia) = \Phi((\frac{1}{2}i1_{\mathscr{A}}) \circ a - a(\frac{1}{2}i1_{\mathscr{A}})^*) = \Phi(\frac{1}{2}i1_{\mathscr{A}}) \circ \Phi(a) - \Phi(a)\Phi(\frac{1}{2}i1_{\mathscr{A}})^* = (\Phi(\frac{1}{2}i1_{\mathscr{A}}) - \Phi(\frac{1}{2}i1_{\mathscr{A}})^*)\Phi(a) = 2\Phi(\frac{1}{2}i1_{\mathscr{A}})\Phi(a)$. This means that, either $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathscr{A}$, or $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathscr{A}$. □

Lemma 2.7. *$\Phi : \mathscr{A} \to \mathscr{B}$ is a $*$-ring isomorphism.*

Proof. Two cases are considered. First case: $\Phi(ia) = i\Phi(a)$, for all elements $a \in \mathscr{A}$. For any elements $a, b \in \mathscr{A}$, we have

$$\begin{aligned}\Phi(ab + b \circ a^*) &= \Phi((ia)(-ib) - (-ib) \circ (ia)^*) = \Phi(ia)\Phi(-ib) \\ &- \Phi(-ib) \circ \Phi(ia)^* = (i\Phi(a))(-i\Phi(b)) - (-i\Phi(b)) \circ (i\Phi(a))^* \\ &= \Phi(a)\Phi(b) + \Phi(b) \circ \Phi(a)^*. \quad (4)\end{aligned}$$

Taking into account (1) and (4) we get $\Phi(ab) = \Phi(a)\Phi(b)$ and $\Phi(b \circ a^*) = \Phi(b) \circ \Phi(a)^*$. It follows that $\Phi(1_{\mathscr{A}}) = 1_{\mathscr{B}}$ and, for any element $a \in \mathscr{A}$, $\Phi(a^*) = \Phi(1_{\mathscr{A}} \circ a^*) = \Phi(1_{\mathscr{A}}) \circ \Phi(a)^* = 1_{\mathscr{B}} \circ \Phi(a)^* = \Phi(a)^*$. As result we obtain that Φ is a $*$-ring isomorphism. Second case: $\Phi(ia) = -i\Phi(a)$, for all elements $a \in \mathscr{A}$. The proof is entirely similar as in the first case.

The Theorem is proved. □

Corollary 2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be type I factor von Neumann algebras acting on complex Hilbert spaces H and K, respectively. Then every bijective mapping $\Phi : \mathscr{A} \to \mathscr{B}$ satisfies $\Phi(ab - b \circ a^*) = \Phi(a)\Phi(b) - \Phi(b) \circ \Phi(a)^*$, for all elements a, b of $\mathscr{A}$, if and only if there is a unitary or conjugate unitary operator $U : H \to K$ such that $\phi(a) = UaU^*$, for all elements $a \in \mathscr{A}$.*

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