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STUDY OF A TYPE OF GENERALIZED BASIC HYPERGEOMETRIC SERIES

Leda Galué
Centro de Investigación de Matemática Aplicada (CIMA)
Facultad de Ingeniería
Universidad del Zulia
Maracaíbo, Venezuela
lgalue@hotmail.com

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Abstract

The aim of this paper is to examine the generalized basic hypergeometric series defined by

$$f(x) = {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, ..., q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, ..., q^{b_r} \end{matrix} ; q, q^n x \right]$$

$M, m_1, m_2, ..., m_r, n \in \mathbb{N}$, $b_1, b_2, ..., b_r \neq 0, -1, -2, ...$, $0 < q < 1$, for this, the effects of the parameters b_j, m_j $(j = 1, 2, \cdots r)$, q and n on the function f are studied, considering some examples and establishing for each one the following characteristics of $f(x)$: the polynomial that defines it, the roots of same and its set of positivily. Also, various graphics which show the behavior of the function are included.

Keywords: Generalized basic hypergeometric series; polynomial; set of positivily; graphic representation.

AMS Subject Classification: 26A33.

1 Introduction

The aim of this paper is to examine the generalized basic hypergeometric series defined by

$$f(x) = {}_{r+1}\phi_r\left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, ..., q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, ..., q^{b_r} \end{matrix} ; q, q^n x\right]$$
$$M, m_1, m_2, ..., m_r, n \in \mathbb{N},\ b_1, b_2, ..., b_r \neq 0, -1, -2, ...,\ 0 < q < 1,$$

for this, the effects of the parameters b_j, m_j $(j = 1, 2, \cdots r)$, q and n on the function f are studied, considering some examples and establishing for each one the following characteristics of $f(x)$: the polynomial that defines it, the roots of same and its set of positivily. Also, various graphics which show the behavior of the function are included.

The results of this study will be used for establish some q-integral inequalities applying the fractional q-integral operator $L_q^n(\cdot)$ [4], which is defined in the following form:

$$L_q^n\{M, b_1, b_2, ..., b_r, \gamma, m_1, m_2, ..., m_r; f(x)\} =$$

$$\frac{x^{-\gamma-1}}{\Gamma_q(M+1)} \int_0^x t^{\gamma}\, {}_{r+1}\phi_r\left[\begin{matrix} q^{-M}, q^{b_1+m_1}, ..., q^{b_r+m_r} \\ q^{b_1}, ..., q^{b_r} \end{matrix} ; q, q^n \frac{t}{x}\right] f(t)\, d_q t,$$

$$M,\ m_1, m_2, ..., m_r \in \mathbb{N}_0,\ n \in \mathbb{N},\ \gamma \in \mathbb{C},$$

$$b_1, b_2, ..., b_r \neq 0, -1, -2, ...,\ 0 < q < 1, \left|\frac{t}{x}\right| < 1.$$

The inequality technique is one of the very useful tools in the study of special functions and theory of approximations [1]. By applying fractional q-integral operators, many researchers have obtained a lot of fractional q-integral inequalities and applications see, for instance, [1], [2], [6]-[8].

Now we present some definitions necessary for the development of the next sections.

1.1 The $q-$shifted factorial

It is defined as: [5, p. 3, No. (1.2.15)]

$$(a;q)_n = \begin{cases} 1, & n = 0. \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & n = 1,2\ldots. \end{cases} \tag{1}$$

1.2 Basic hypergeometric series

This series was introduced by Heine and it is defined as: [5]

$$\phi(a,b;c;q,z) = {}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \tag{2}$$

where it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \ldots,$ and $(a;q)_n$ is the q-shifted factorial defined in (1).

1.3 Generalized basic hypergeometric series

A generalization of the basic hypergeometric series ${}_2\phi_1$, is given by: [5]

$$ {}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z) = $$

$$\sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{3}$$

where $b_1, ..., b_s \neq q^{-m}$ for $m = 0, 1, 2, ...;$ $\binom{n}{2} = \frac{n(n-1)}{2}$; $q \neq 0$ when $r > s+1$ and $\lim_{q\to 1} {}_r\phi_s = {}_rF_s$.

As particular case of (3) we have

$$ {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, ..., q^{b_r+m_r} \\ q^{b_1}, q^{b_2}, ..., q^{b_r} \end{matrix} ; q, q^n x \right] = $$

$$\sum_{k=0}^{M} \frac{(q^{-M};q)_k \; (q^{b_1+m_1};q)_k \cdots (q^{b_r+m_r};q)_k}{(q;q)_k \; (q^{b_1};q)_k \cdots (q^{b_r};q)_k} (q^n x)^k, \tag{4}$$

$M,\ m_1, m_2, ..., m_r,\ n \in \mathbb{N},\ b_1, b_2, ..., b_r \neq 0, -1, -2, ...,\ 0 < q < 1,$
where we use the result [5, p. 4, No. (1.2.23)]

$$\left(q^{-m}; q\right)_n = 0 \quad n = m{+}1, m{+}2, \ldots, \text{ with } m = 0, 1, 2, \ldots, \text{ and } q \neq 0,$$

then the $q-$hypergeometric series given in (4) is terminating, which is in fact a polynomial of x [3].

Let be $f(x) \equiv {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \cdots, q^{b_r+m_r} \\ q^{b_1}, \cdots, q^{b_r} \end{matrix} ; q, q^n x \right]$. From (4) applying the definition (1) we can write

$$f(x) \equiv {}_{r+1}\phi_r \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \cdots, q^{b_r+m_r} \\ q^{b_1}, \cdots, q^{b_r} \end{matrix} ; q, q^n x \right] = 1 +$$

$$\frac{\left(1-q^{-M}\right)\left(1-q^{b_1+m_1}\right)\cdots\left(1-q^{b_r+m_r}\right)}{(1-q)\left(1-q^{b_1}\right)\cdots\left(1-q^{b_r}\right)} \left(q^n x\right) +$$
$$\frac{\left(1-q^{-M}\right)\left(1-q^{-M+1}\right)\left(1-q^{b_1+m_1}\right)\left(1-q^{b_1+m_1+1}\right)}{(1-q)\left(1-q^2\right)\left(1-q^{b_1}\right)\left(1-q^{b_1+1}\right)} \cdots \times$$
$$\frac{\left(1-q^{b_r+m_r}\right)\left(1-q^{b_r+m_r+1}\right)}{\left(1-q^{b_r}\right)\left(1-q^{b_r+1}\right)} \left(q^n x\right)^2 + \overset{M-3}{\cdots} +$$
$$\frac{\left(1-q^{-M}\right)\left(1-q^{-M+1}\right)\cdots\left(1-q^{-1}\right)}{(1-q)\left(1-q^2\right)\cdots\left(1-q^M\right)} \times$$
$$\frac{\left(1-q^{b_1+m_1}\right)\left(1-q^{b_1+m_1+1}\right)\cdots\left(1-q^{b_1+m_1+M-1}\right)}{\left(1-q^{b_1}\right)\left(1-q^{b_1+1}\right)\cdots\left(1-q^{b_1+M-1}\right)} \cdots \times$$
$$\frac{\left(1-q^{b_r+m_r}\right)\left(1-q^{b_r+m_r+1}\right)\cdots\left(1-q^{b_r+m_r+M-1}\right)}{\left(1-q^{b_r}\right)\left(1-q^{b_r+1}\right)\cdots\left(1-q^{b_r+M-1}\right)} \left(q^n x\right)^M . \qquad (5)$$

The following results will be very useful for determine the sign of the factors that appear in (5).

$$\left(1-q^A\right) > 0, \quad \text{if } 0 < q < 1,\ A > 0. \qquad (6)$$

$$(1-q^A) = 1 - \frac{1}{q^{-A}} = -\frac{(1-q^{-A})}{q^{-A}} < 0, \quad \text{if } 0<q<1,\ A<0. \quad (7)$$

Note: Observe that
i) $f(0) = 1$.
ii) $(q^{-M};q)_h = (1-q^{-M})(1-q^{-M+1})\cdots(1-q^{-M+h-1})$, $h = 1,2,\ldots M$, then according to (7) all the factors of this expression are negative.
iii) $(q;q)_h = (1-q)(1-q^2)\cdots(1-q^h)$, $h = 1,2,\ldots M$, then according to (6) all the factors of this expression are positive.

2 Effect of the parameters $b_j,\ j = 1,2,\ldots,r$ on the function ${}_{r+1}\phi_r\left[\cdot\ ;q,q^n x\right]$

In this section the effect of the parameters $b_j,\ j = 1,2,\ldots,r$ on the function $f(x) \equiv {}_{r+1}\phi_r\left[\begin{matrix} q^{-M}, q^{b_1+m_1}, \cdots, q^{b_r+m_r} \\ q^{b_1}, \cdots, q^{b_r} \end{matrix} ;q,q^n x\right]$, where $0<q<1$, is studied.

The parameters $b_j,\ j = 1,2,\ldots,r$ influence in the sign and the value of the factors of

$$(q^{b_j};q)_h = \left(1-q^{b_j}\right)\left(1-q^{b_j+1}\right)\cdots\left(1-q^{b_j+h-1}\right)$$

and

$$(q^{b_j+m_j};q)_h = \left(1-q^{b_j+m_j}\right)\left(1-q^{b_j+m_j+1}\right)\cdots\left(1-q^{b_j+m_j+h-1}\right),$$

where $h = 1,2,\ldots,M$.

Now we present two examples that show the effect of $b_j,\ j = 1,2,\ldots,r$ on the factors of $(q^{b_j};q)_h$.

1) Let be $b_j = 3.5, M = 4$.

From (1) we have

$$(q^{3.5};q)_h = \left(1-q^{3.5}\right)\cdots\left(1-q^{2.5+h}\right),\ h = 1,2,3,4.$$

Table 1. Factors and sign of $(q^{3.5};q)_h$ for different values of h

h	$(q^{3.5};q)_h$	Sign of $(q^{3.5};q)_h$
1	$(1-q^{3.5})$	+
2	$(1-q^{3.5})(1-q^{4.5})$	+
3	$(1-q^{3.5})(1-q^{4.5})(1-q^{5.5})$	+
4	$(1-q^{3.5})(1-q^{4.5})(1-q^{5.5})(1-q^{6.5})$	+

Therefore, according to (6), all the factors of $(q^{3.5};q)_h,\ h=1,2,3,4,$ are positive.

2) Let be $b_j=-2.3, M=5.$

From (1)

$$(q^{-2.3};q)_h=\left(1-q^{-2.3}\right)\left(1-q^{-1.3}\right)\cdots\left(1-q^{-3.3+h}\right),\ h=1,2,3,4,5.$$

Table 2. Factors and sign of $(q^{-2.3};q)_h$ for different values of h

h	$(q^{-2.3};q)_h$	Sign of $(q^{-2.3};q)_h$
1	$(1-q^{-2.3})$	−
2	$(1-q^{-2.3})(1-q^{-1.3})$	+
3	$(1-q^{-2.3})(1-q^{-1.3})(1-q^{-0.3})$	−
4	$(1-q^{-2.3})(1-q^{-1.3})(1-q^{-0.3})(1-q^{0.7})$	−
5	$(1-q^{-2.3})(1-q^{-1.3})(1-q^{-0.3})(1-q^{0.7})(1-q^{1.7})$	−

According to (7) and (6), all factors of $(q^{-2.3};q)_h,\ h=1,2,3,$ are negative, whereas $(q^{-2.3};q)_h,\ h=4,5,$ have 3 negative factors and $h-3$ positive factors.

For the general analysis we shall consider the following factors with their respective conditions, in all cases $j=1,2,\cdots,r;\ h=1,2,\cdots,M.$

Table 3. Conditions for determinate the possibles values of b_j

	Factor	Condition
i)	$\left(1-q^{b_j+h-1}\right)$	$b_j+h-1\neq 0.$
ii)	$\left(1-q^{b_j+m_j+h-1}\right)$	$b_j+m_j+h-1\neq 0.$

2.1 Example: Let be $f(x) = {}_2\phi_1\left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix} ; q, q^n x\right]$,
$q = 0.8,\ M = 3,\ m = 5,\ n = 2.$

From (5) we have

$$f(x) = {}_2\phi_1\left[\begin{matrix} 0.8^{-3}, 0.8^{b+5} \\ 0.8^b \end{matrix} ; 0.8, 0.8^2 x\right] = 1+$$

$$\frac{\left(1-0.8^{-3}\right)\left(1-0.8^{b+5}\right)}{\left(1-0.8\right)\left(1-0.8^{b}\right)}\left(0.8^2 x\right) + \frac{\left(1-0.8^{-3}\right)\left(1-0.8^{-2}\right)}{\left(1-0.8\right)\left(1-0.8^{2}\right)} \times$$
$$\frac{\left(1-0.8^{b+5}\right)\left(1-0.8^{b+6}\right)}{\left(1-0.8^{b}\right)\left(1-0.8^{b+1}\right)}\left(0.8^2 x\right)^2 + \frac{\left(1-0.8^{-3}\right)\left(1-0.8^{-2}\right)}{\left(1-0.8\right)\left(1-0.8^{2}\right)} \times$$
$$\frac{\left(1-0.8^{-1}\right)}{\left(1-0.8^{3}\right)}\frac{\left(1-0.8^{b+5}\right)\left(1-0.8^{b+6}\right)\left(1-0.8^{b+7}\right)}{\left(1-0.8^{b}\right)\left(1-0.8^{b+1}\right)\left(1-0.8^{b+2}\right)}\left(0.8^2 x\right)^3,$$

with the following conditions: (see Table 3)

i) $b \neq 0,\ b \neq -1;\ b \neq -2.$ ii) $b \neq -5,\ b \neq -6;\ b \neq -7.$

Therefore the possible values of b are:
$b < -7,\ -7 < b < -6,\ -6 < b < -5,\ -5 < b < -2,\ -2 < b < -1,\ -1 < b < 0,\ b > 0.$

Now we present the graphic representation of $f(x)$ for different values of b.

Case 1. $b < -7$:
$b = -9.5$: $f(x) = {}_2\phi_1\left[\begin{matrix} 0.8^{-3}, 0.8^{-4.5} \\ 0.8^{-9.5} \end{matrix} ; 0.8, 0.8^2 x\right] = 0.15040x^2 - 0.71967x - 8.5037 \times 10^{-3}x^3 + 1.$
Roots: 10.925, 4.1963, 2.5652.

$b = -8.5$: $f(x) = {}_2\phi_1\left[\begin{matrix} 0.8^{-3}, 0.8^{-3.5} \\ 0.8^{-8.5} \end{matrix} ; 0.8, 0.8^2 x\right] = 0.10992x^2 - 0.63739x - 4.3881 \times 10^{-3}x^3 + 1.$

Roots: 17.489, 4.9024, 2.6580.

$$b=-8.0:\ f(x)={}_2\phi_1\left[\begin{matrix}0.8^{-3},0.8^{-3.0}\\0.8^{-8.0}\end{matrix};0.8,0.8^2x\right]=8.7477\times10^{-2}x^2-$$
$0.58604x-2.5474\times10^{-3}x^3+1.$
Roots: 26.102, 5.5059, 2.7315.

$$b=-7.5:\ f(x)={}_2\phi_1\left[\begin{matrix}0.8^{-3},0.8^{-2.5}\\0.8^{-7.5}\end{matrix};0.8,0.8^2x\right]=6.4044\times10^{-2}x^2-$$
$0.52598x-1.0276\times10^{-3}x^3+1.$
Roots: 53.017, 6.4728, 2.8359.

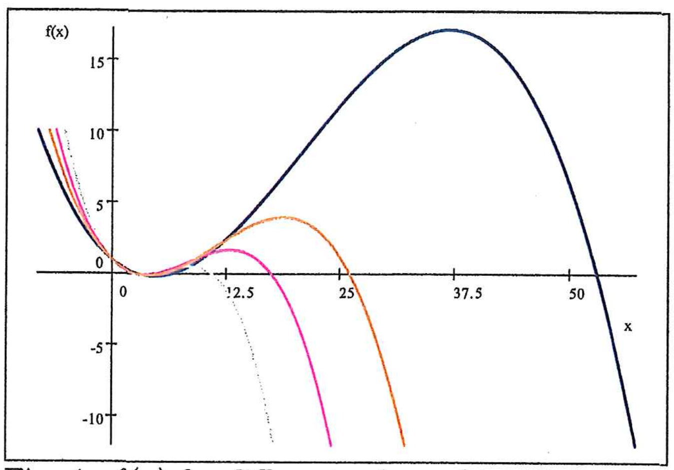


Fig. 1. $f(x)$ for different values of b with $b<-7$
gray $b=-9.5$ magenta $b=-8.5$ sienna $b=-8.0$ blue $b=-7.5$

Case 2. $-7<b<-6$:

$$b=-6.9:\ f(x)={}_2\phi_1\left[\begin{matrix}0.8^{-3},0.8^{-1.9}\\0.8^{-6.9}\end{matrix};0.8,0.8^2x\right]=3.5811\times10^{-2}x^2-$$
$0.43964x+1.3057\times10^{-4}x^3+1.$
Roots: 8.8299, 3.0314, -286.14.

$$b=-6.8:\ f(x)={}_2\phi_1\left[\begin{matrix}0.8^{-3},0.8^{-1.8}\\0.8^{-6.8}\end{matrix};0.8,0.8^2x\right]=3.1252\times10^{-2}x^2-$$
$0.42346x+2.3311\times10^{-4}x^3+1.$

Roots: 9.5108, 3.0757, −146.65.

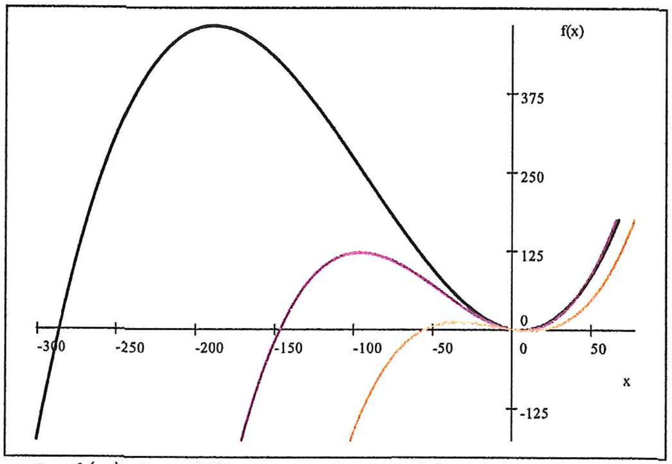


Fig. 2. $f(x)$ for different values of b with $-7 < b < -6$
black $b = -6.9$ purple $b = -6.8$ sienna $b = -6.3$

$b = -6.3: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-1.3} \\ 0.8^{-6.3} \end{matrix} ; 0.8, 0.8^2 x \right] = 1.0200 \times 10^{-2} x^2 - 0.33339x + 3.0030 \times 10^{-4} x^3 + 1.$
Roots: 17.829, 3.3849, −55.179.

Case 3. $-6 < b < -5$:
$b = -5.9 : \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.9} \\ 0.8^{-5.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 2.7627 \times 10^{-3} x^2 - 1.4209 \times 10^{-4} x^3 - 0.24844x.$
Roots: $-11.637 + 41.258i$, $-11.637 - 41.258i$, 3.8299.

$b = -5.5 : \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.5} \\ 0.8^{-5.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 9.1106 \times 10^{-3} x^2 - 7.1786 \times 10^{-4} x^3 - 0.14926x.$
Roots: 4.7800, $-8.7357 + 14.667i$, $-8.7357 - 14.667i$.

$b = -5.1 : \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{-0.1} \\ 0.8^{-5.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 1 - 3.9459 \times 10^{-3} x^2 - 4.4832 \times 10^{-4} x^3 - 3.2455 \times 10^{-2} x.$

Roots: 9.2832, $-9.0424 + 12.59i$, $-9.0424 - 12.59i$.

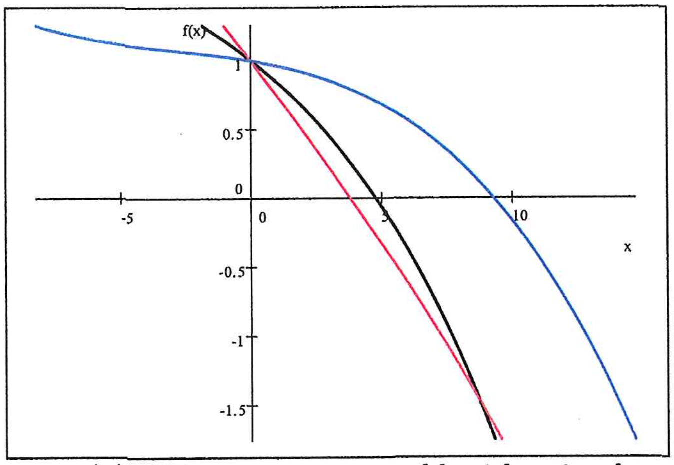


Fig. 3. $f(x)$ for different values of b with $-6 < b < -5$
red $b = -5.9$ black $b = -5.5$ blue $b = -5.1$

Case 4. $-5 < b < -2$:

$b = -4.9$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{0.1} \\ 0.8^{-4.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 3.3917 \times 10^{-2} x + 5.3203 \times 10^{-3} x^2 + 7.1713 \times 10^{-4} x^3 + 1.$

Roots: -12.528, $2.5546 + 10.236i$, $2.5546 - 10.236i$.

$b = -4.5$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{0.5} \\ 0.8^{-4.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.18617x + 4.4741 \times 10^{-2} x^2 + 8.3971 \times 10^{-3} x^3 + 1.$

Roots: -5.347, $9.4493 \times 10^{-3} + 4.7193i$, $9.4493 \times 10^{-3} - 4.7193i$.

$b = -4.0$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{1.0} \\ 0.8^{-4.0} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.42320x + 0.15984x^2 + 4.5467 \times 10^{-2} x^3 + 1.$

Roots: -2.9126, $-0.30153 + 2.7314i$, $-0.30153 - 2.7314i$.

$b = -3.5$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{1.5} \\ 0.8^{-3.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 0.73298x + 0.41958x^2 + 0.18757x^3 + 1.$

Roots: -1.7469, $-0.24497 - 1.7297i$, $-0.24497 + 1.7297i$.

$b = -3.0:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{2.0} \\ 0.8^{-3.0} \end{matrix} ; 0.8, 0.8^2 x \right] = 1.152x + 0.99942x^2 + 0.77385x^3 + 1.$

Roots: $-0.12186 + 1.1038i$, $-0.12186 - 1.1038i$, -1.0478.

$b = -2.1:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{2.9} \\ 0.8^{-2.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 2.431x + 5.0783x^2 + 49.062x^3 + 1.$

Roots: $7.0043 \times 10^{-2} + 0.28066i$, $7.0043 \times 10^{-2} - 0.28066i$, -0.24359.

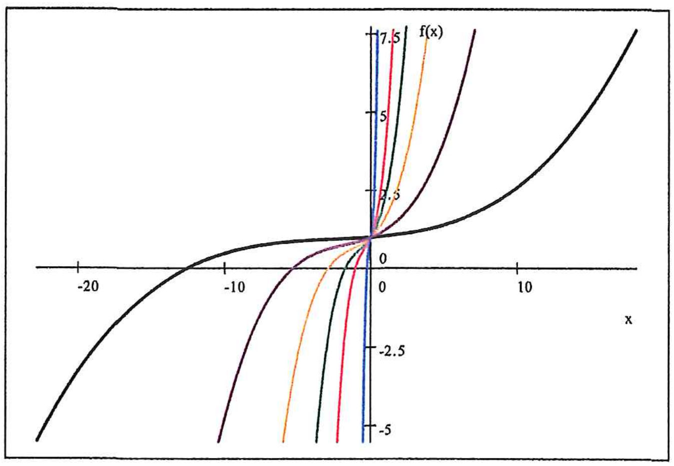


Fig. 4. $f(x)$ for different values of b with $-5 < b < -2$
black $b = -4.9$ purple $b = -4.5$ sienna -4.0 green $b = -3.5$
red -3.0 blue $b = -2.1$

Case 5. $-2 < b < -1:$

$b = -1.9:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.1} \\ 0.8^{-1.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 2.8841x + 7.7730x^2 - 78.481x^3 + 1.$

Roots: 0.32874, $-0.11485 + 0.15991i$, $-0.11485 - 0.15991i$.

$b=-1.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.5} \\ 0.8^{-1.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 4.1587x + 22.325x^2 - 49.013x^3 + 1.$

Roots: 0.63845, $-9.1476 \times 10^{-2} - 0.15359i$, $-9.1476 \times 10^{-2} + 0.15359i$.

$b=-1.1: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.9} \\ 0.8^{-1.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 6.3713x + 187.74x^2 - 247.62x^3 + 1.$

Roots: $-1.9325 \times 10^{-2} - 6.8517 \times 10^{-2} i$, $-1.9325 \times 10^{-2} + 6.8517 \times 10^{-2} i$, 0.79683.

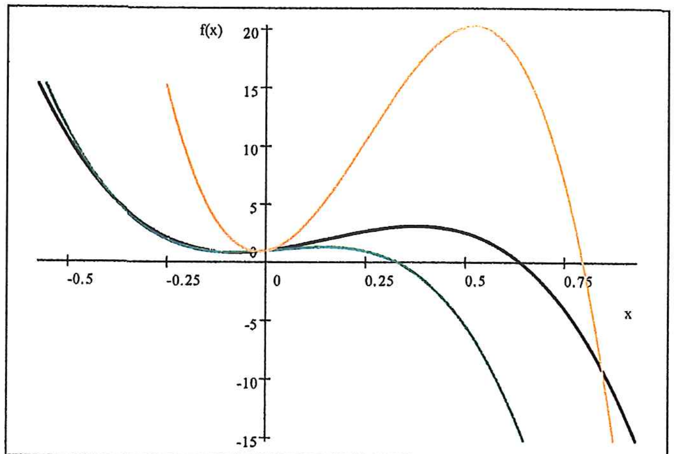


Fig. 5. $f(x)$ for different values of b with $-2 < b < -1$
green $b=-1.9$ black $b=-1.5$ sienna $b=-1.1$

Case 6. $-1 < b < 0$:

$b=-0.9: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.1} \\ 0.8^{-0.9} \end{matrix} ; 0.8, 0.8^2 x \right] = 8.2201x - 253.14x^2 + 283.56x^3 + 1.$

Roots: 0.85391, 8.6519×10^{-2}, -4.7734×10^{-2}.

$b=-0.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.5} \\ 0.8^{-0.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 16.373x - 109.64x^2 + 96.738x^3 + 1.$

Roots: 0.942, 0.23752, −0.0462.

$b = -0.1: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{4.9} \\ 0.8^{-0.1} \end{matrix} ; 0.8, 0.8^2 x \right] = 89.874x - 361.55x^2 + 269.47x^3 + 1.$

Roots: $1.0067, 0.34562, -1.0665 \times 10^{-2}$.

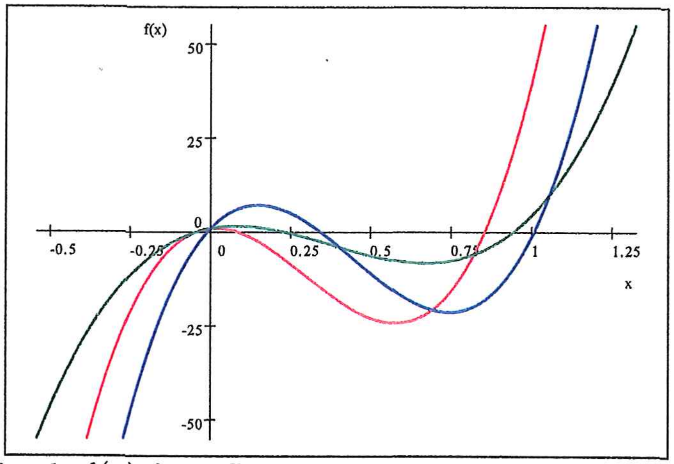


Fig. 6. $f(x)$ for different values of b with $-1 < b < 0$
red $b = -0.9$ green $b = -0.5$ blue $b = -0.1$

Case 7. $b > 0$:

$b = 0.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{5.5} \\ 0.8^{0.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 54.962x^2 - 20.423x - 34.24x^3 + 1.$

Roots: $1.0761, 0.47148, 5.7562 \times 10^{-2}$.

$b = 1.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{6.5} \\ 0.8^{1.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 15.596x^2 - 8.2081x - 8.0181x^3 + 1.$

Roots: 1.1482, 0.6225, 0.17450.

$b = 3.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{8.5} \\ 0.8^{3.5} \end{matrix} ; 0.8, 0.8^2 x \right] = 6.6414x^2 - 4.7824x - 2.7845x^3 + 1.$

Roots: 1.2143, 0.80215, 0.36870.

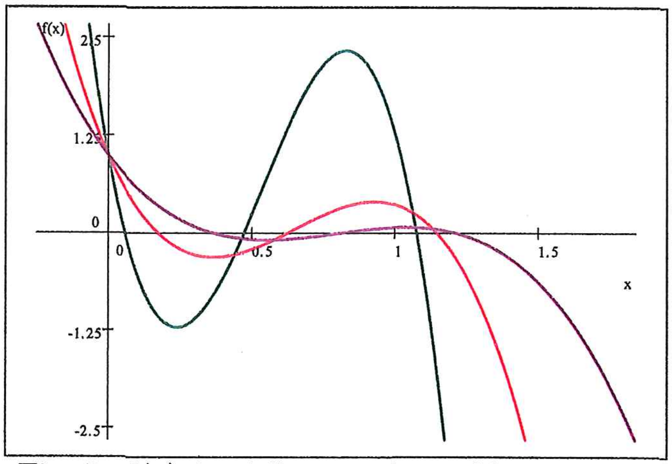


Fig. 7. $f(x)$ for different values of b with $b > 0$
green $b = 0.5$ red $b = 1.5$ purple $b = 3.5$

The previous results are compiled in the following Table.

Table 4. Characteristics of $f(x)$ for different values of b

Case	Polynomial	Roots	Set of Positivity
1 and 7	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
2	$a_2x^2 - a_1x + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
3	$1 - a_2x^2 - a_3x^3 - a_1x$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
4	$a_1x + a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 ir	$(r_1; +\infty)$.
5	$a_1x + a_2x^2 - a_3x^3 + 1$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
6	$a_1x - a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.

In this Table and through of all paper is used the following notation:

nrr: negative real root. prr: positive real root.

ir: imaginary roots. $a_1, a_2, a_3, \cdots$ positive constants.

$r_1 < r_2 < r_3 < \cdots$

2.2 Example: $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x \right]$,

$q = 0.6,\ M = 2,\ m_1 = 2,\ m_2 = 3,\ n = 1.$

From (5) we can write

$$f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{b_1+2}, 0.6^{b_2+3} \\ 0.6^{b_1}, 0.6^{b_2} \end{matrix} ; 0.6, 0.6x \right] = 1 + \frac{(1-0.6^{-2})}{(1-0.6)} \times$$

$$\frac{\left(1-0.6^{b_1+2}\right)\left(1-0.6^{b_2+3}\right)}{\left(1-0.6^{b_1}\right)\left(1-0.6^{b_2}\right)}(0.6x) + \frac{(1-0.6^{-2})(1-0.6^{-1})}{(1-0.6)(1-0.6^2)} \times$$

$$\frac{\left(1-0.6^{b_1+2}\right)\left(1-0.6^{b_1+3}\right)\left(1-0.6^{b_2+3}\right)\left(1-0.6^{b_2+4}\right)}{\left(1-0.6^{b_1}\right)\left(1-0.6^{b_1+1}\right)\left(1-0.6^{b_2}\right)\left(1-0.6^{b_2+1}\right)}(0.6x)^2 .$$

Using Table 3 we establish the following conditions:

i) $b_1 \neq 0, -1$. ii) $b_1 \neq -2, -3$. i) $b_2 \neq 0, -1$. ii) $b_2 \neq -3, -4$.

Therefore the possible values of b_1 y b_2 are:

$b_1 < -3,\ -3 < b_1 < -2,\ -2 < b_1 < -1,\ -1 < b_1 < 0,\ b_1 > 0.$

$b_2 < -4,\ -4 < b_2 < -3,\ -3 < b_2 < -1,\ -1 < b_2 < 0,\ b_2 > 0.$

We consider the following cases:

Case 1.1. $b_1 < -3,\ b_2 < -4$:

$b_1 = -5.8,\ b_2 = -6.2:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-3.8}, 0.6^{-3.2} \\ 0.6^{-5.8}, 0.6^{-6.2} \end{matrix}; 0.6, 0.6x\right] =$
$4.6221 \times 10^{-3}x^2 - 0.15738x + 1.$
Roots: 25.597, 8.4522.

Case 1.2. $b_1 < -3,\ -4 < b_2 < -3:$
$b_1 = -8.4,\ b_2 = -3.3:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-6.4}, 0.6^{-0.3} \\ 0.6^{-8.4}, 0.6^{-3.3} \end{matrix}; 0.6, 0.6x\right] =$
$1 - 1.0219 \times 10^{-3}x^2 - 3.5271 \times 10^{-2}x.$
Roots: 18.469, −52.985.

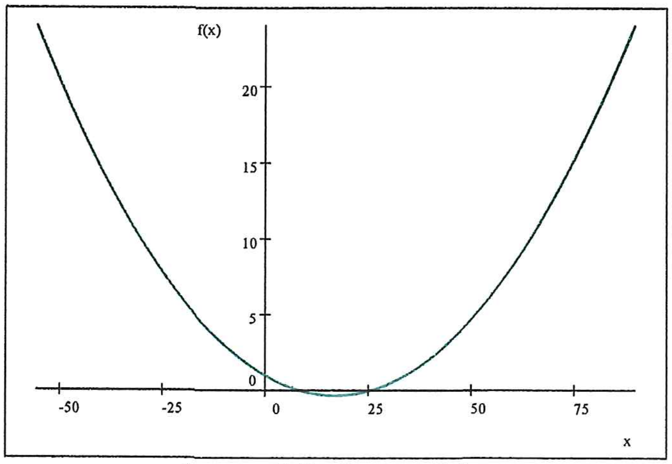
Case 1.3. $b_1 < -3,\ -3 < b_2 < -1:$
$b_1 = -6.1,\ b_2 = -2.3:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-4.1}, 0.6^{0.7} \\ 0.6^{-6.1}, 0.6^{-2.3} \end{matrix}; 0.6, 0.6x\right] =$
$0.11833x + 1.4067 \times 10^{-2}x^2 + 1.$
Roots: $-4.206 + 7.3074i,\ -4.206 - 7.3074i.$

$b_1 = -3.2,\ b_2 = -1.4:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-1.2}, 0.6^{1.6} \\ 0.6^{-3.2}, 0.6^{-1.4} \end{matrix}; 0.6, 0.6x\right] =$
$0.29217x + 3.0669 \times 10^{-2}x^2 + 1.$
Roots: $-4.7633 + 3.1492i,\ -4.7633 - 3.1492i.$

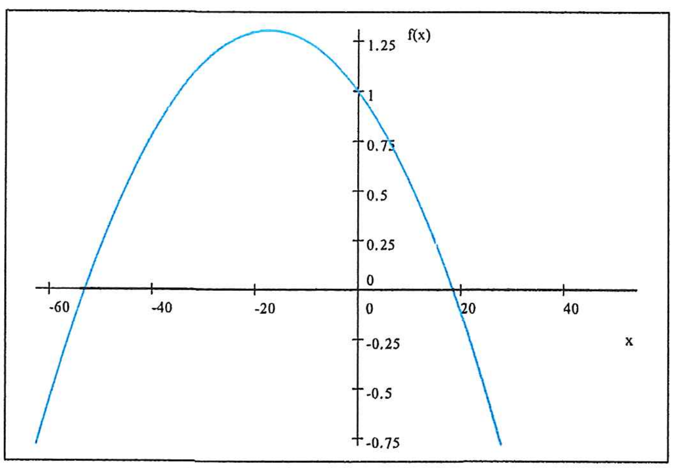
Case 1.4. $b_1 < -3,\ -1 < b_2 < 0:$
$b_1 = -7.6,\ b_2 = -0.5:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-5.6}, 0.6^{2.5} \\ 0.6^{-7.6}, 0.6^{-0.5} \end{matrix}; 0.6, 0.6x\right] =$
$2.2901x - 1.7832x^2 + 1.$
Roots: 1.6286, −0.34434.

Case 1.5. $b_1 < -3,\ b_2 > 0:$
$b_1 = -4.7,\ b_2 = 3.9:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-2.7}, 0.6^{6.9} \\ 0.6^{-4.7}, 0.6^{3.9} \end{matrix}; 0.6, 0.6x\right] =$
$0.14609x^2 - 0.88770x + 1.$
Roots: 4.5827, 1.4937.

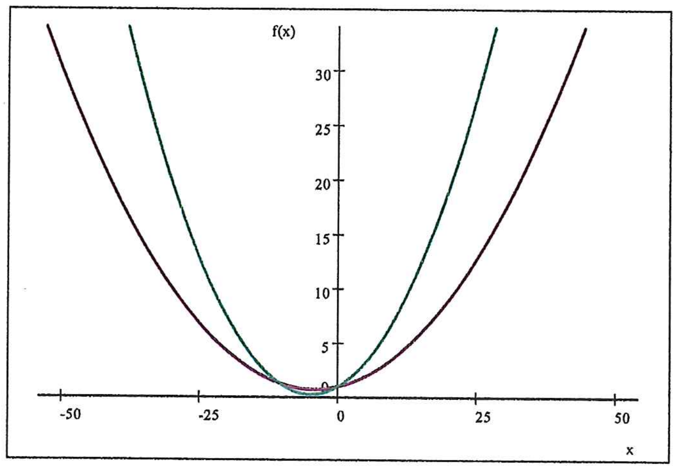
Fig. 8. Case 1.1-Case 1.5: $f(x)$ for different values of b_1, b_2



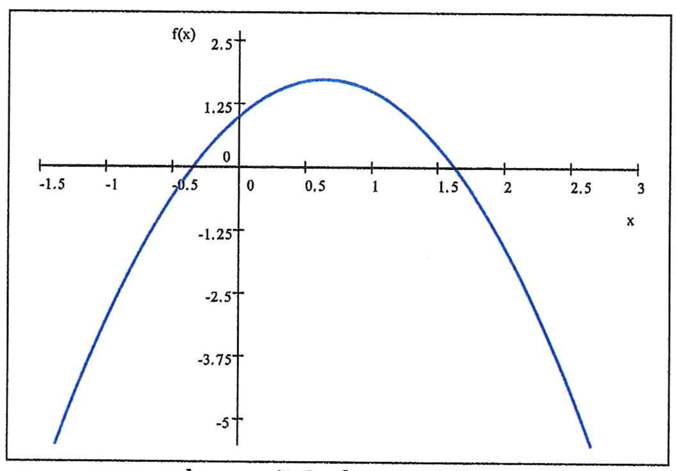
$b_1 = -5.8,\ b_2 = -6.2$



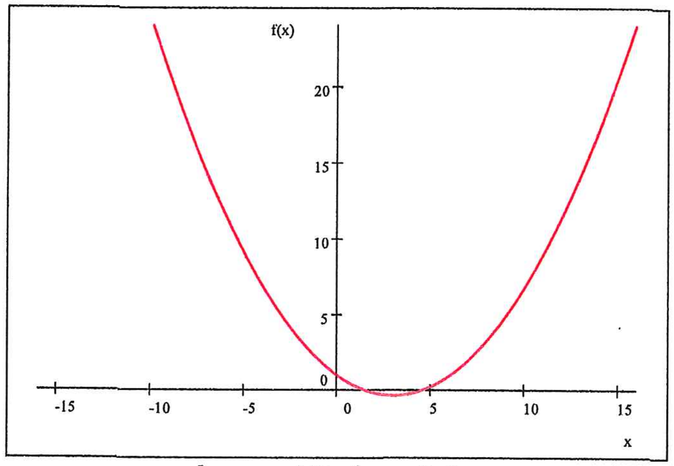
$b_1 = -8.4,\ b_2 = -3.3$



purple $b_1 = -6.1,\ b_2 = -2.3$ green $b_1 = -3.2,\ b_2 = -1.4$



$b_1 = -7.6,\ b_2 = -0.5$



$b_1 = -4.7,\ b_2 = 3.9$

Case 2.1. $-3 < b_1 < -2,\ b_2 < -4:$

$$b_1 = -2.8,\ b_2 = -4.2:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.8}, 0.6^{-1.2} \\ 0.6^{-2.8}, 0.6^{-4.2} \end{matrix};0.6, 0.6x\right] =$$

$1 - 4.9779 \times 10^{-5}x^2 - 4.7455 \times 10^{-2}x.$

Roots: $20.626,\ -973.95.$

Case 2.2. $-3 < b_1 < -2,\ -4 < b_2 < -3:$

$$b_1 = -2.5,\ b_2 = -3.2:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.5}, 0.6^{-0.2} \\ 0.6^{-2.5}, 0.6^{-3.2} \end{matrix};0.6, 0.6x\right] =$$

$1.5452 \times 10^{-4}x^2 - 7.8196 \times 10^{-3}x + 1.$

Roots: $25.302 + 76.363i,\ 25.302 - 76.363i.$

Case 2.3. $-3 < b_1 < -2,\ -3 < b_2 < -1:$

$$b_1 = -2.1,\ b_2 = -1.6:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.1}, 0.6^{1.4} \\ 0.6^{-2.1}, 0.6^{-1.6} \end{matrix};0.6, 0.6x\right] =$$

$2.9360 \times 10^{-2}x - 1.7669 \times 10^{-2}x^2 + 1.$

Roots: $8.3997,\ -6.738.$

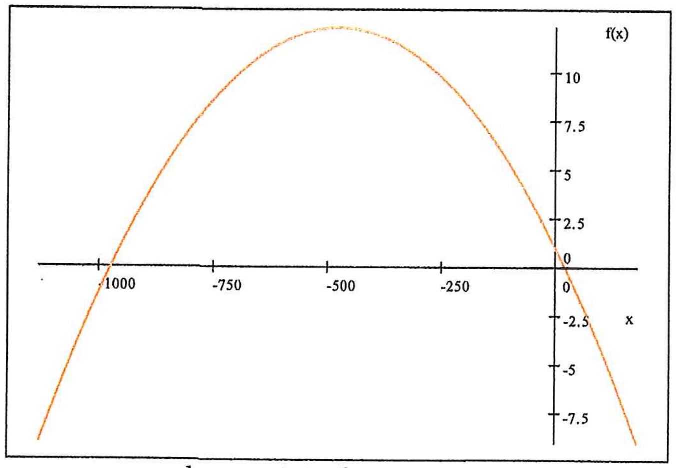
$$b_1 = -2.4,\ b_2 = -2.6:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.4}, 0.6^{0.4} \\ 0.6^{-2.4}, 0.6^{-2.6} \end{matrix}; 0.6, 0.6x\right] =$$
$1.6729 \times 10^{-2}x - 1.0677 \times 10^{-3}x^2 + 1.$
Roots: $39.426,\ -23.757.$

Case 2.4. $-3 < b_1 < -2,\ -1 < b_2 < 0:$
$$b_1 = -2.9,\ b_2 = -0.4:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.9}, 0.6^{2.6} \\ 0.6^{-2.9}, 0.6^{-0.4} \end{matrix}; 0.6, 0.6x\right] =$$
$1.4846x + 8.9799 \times 10^{-2}x^2 + 1.$
Roots: $-0.70350,\ -15.829.$

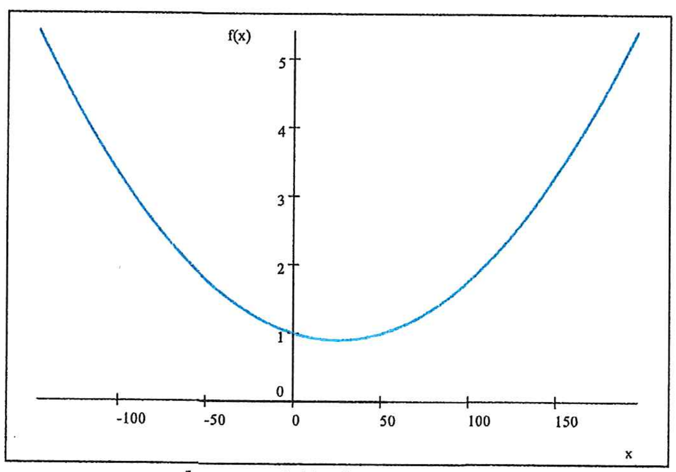
$$b_1 = -2.5,\ b_2 = -0.4:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.5}, 0.6^{2.6} \\ 0.6^{-2.5}, 0.6^{-0.4} \end{matrix}; 0.6, 0.6x\right] =$$
$0.97288x + 0.37915x^2 + 1.$
Roots: $-1.2830 + 0.99573i,\ -1.2830 - 0.99573i.$

Case 2.5. $-3 < b_1 < -2,\ b_2 > 0:$
$$b_1 = -2.3,\ b_2 = 5.7:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{-0.3}, 0.6^{8.7} \\ 0.6^{-2.3}, 0.6^{5.7} \end{matrix}; 0.6, 0.6x\right] =$$
$1 - 4.2196 \times 10^{-2}x^2 - 0.20625x.$
Roots: $3.0032,\ -7.8911.$

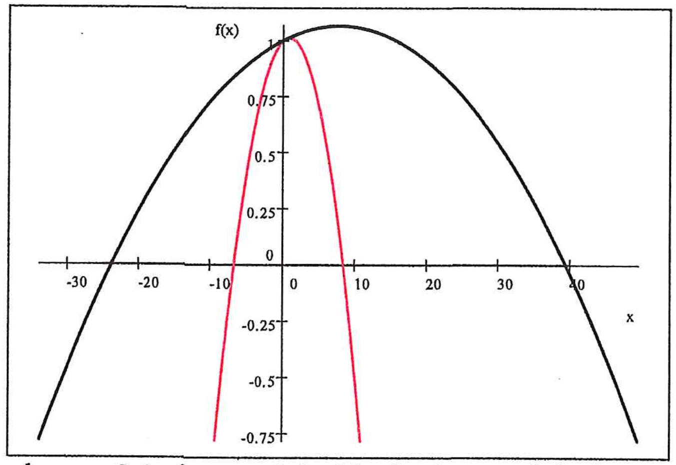
Fig. 9. Case 2.1-Case 2.5: $f(x)$ for different values of b_1, b_2



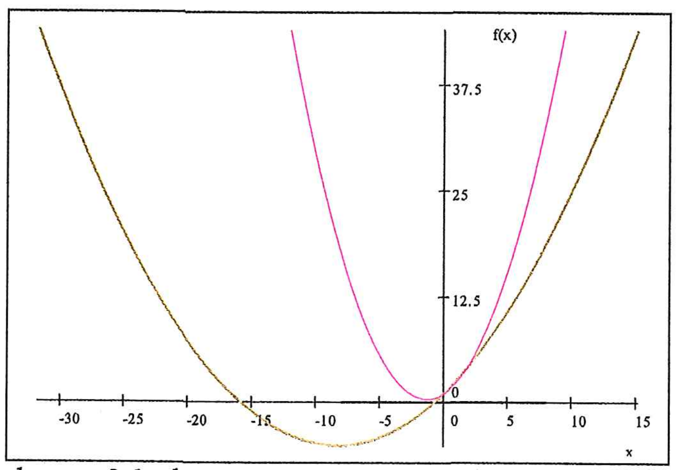
$b_1 = -2.8,\ b_2 = -4.2$



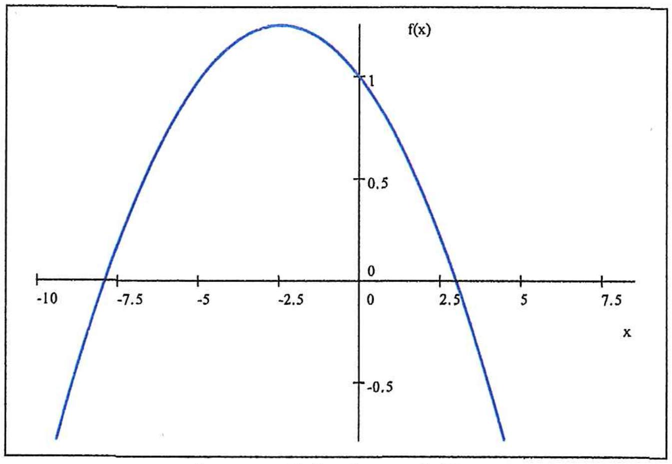
$b_1 = -2.5,\ b_2 = -3.2$



red $b_1 = -2.1,\ b_2 = -1.6$ black $b_1 = -2.4,\ b_2 = -2.6$



green $b_1 = -2.9,\ b_2 = -0.4$ magenta $b_1 = -2.5,\ b_2 = -0.4$



$b_1 = -2.3,\ b_2 = 5.7$

Case 3.1. $-2 < b_1 < -1,\ b_2 < -4:$

$$b_1 = -1.1,\ b_2 = -7.6:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.9}, 0.6^{-4.6} \\ 0.6^{-1.1}, 0.6^{-7.6} \end{matrix}; 0.6, 0.6x\right] =$$

$0.26004x + 0.36235x^2 + 1.$

Roots: $-0.35883 + 1.622i,\ -0.35883 - 1.622i.$

Case 3.2. $-2 < b_1 < -1,\ -4 < b_2 < -3:$

$$b_1 = -1.5,\ b_2 = -3.6:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.5}, 0.6^{-0.6} \\ 0.6^{-1.5}, 0.6^{-3.6} \end{matrix}; 0.6, 0.6x\right] =$$

$3.5385 \times 10^{-2}x - 0.00271x^2 + 1.$

Roots: $26.817,\ -13.760.$

Case 3.3. $-2 < b_1 < -1,\ -3 < b_2 < -1:$

$$b_1 = -1.9,\ b_2 = -2.4:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.1}, 0.6^{0.6} \\ 0.6^{-1.9}, 0.6^{-2.4} \end{matrix}; 0.6, 0.6x\right] =$$

$2.1857 \times 10^{-3}x^2 - 8.8818 \times 10^{-3}x + 1.$

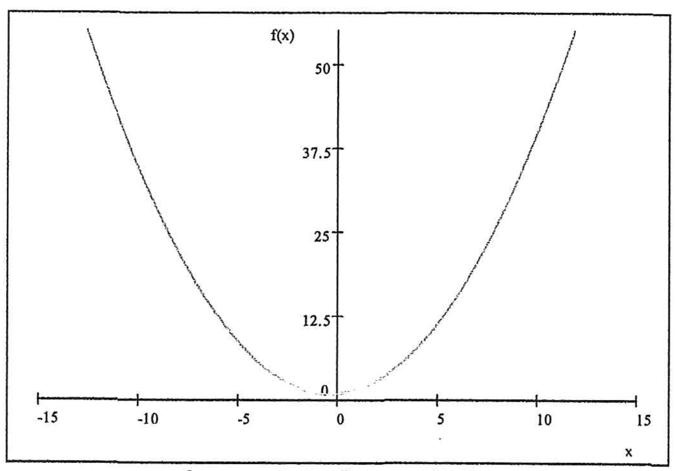
Roots: $2.0318 + 21.293i,\ 2.0318 - 21.293i.$

$b_1 = -1.7,\ b_2 = -1.4:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.3}, 0.6^{1.6} \\ 0.6^{-1.7}, 0.6^{-1.4} \end{matrix}; 0.6, 0.6x\right] =$
$0.335x^2 - 0.14645x + 1.$
Roots: $0.218\,58 + 1.7139i,\ 0.218\,58 - 1.7139i.$

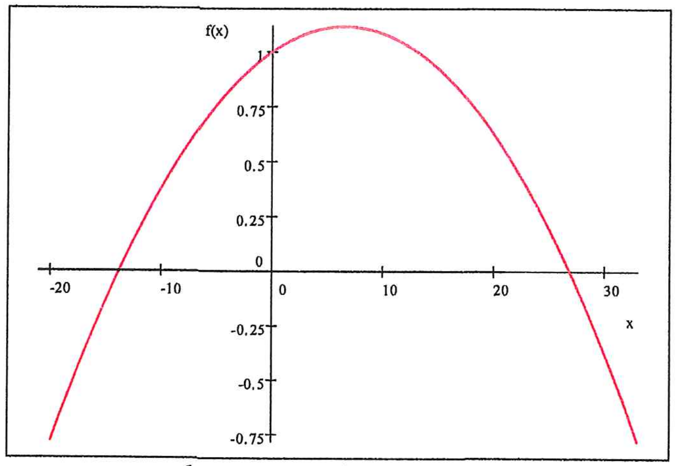
Case 3.4. $-2 < b_1 < -1,\ -1 < b_2 < 0:$
$b_1 = -1.4,\ b_2 = -0.2:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.6}, 0.6^{2.8} \\ 0.6^{-1.4}, 0.6^{-0.2} \end{matrix}; 0.6, 0.6x\right] =$
$1 - 18.734x^2 - 4.7665x.$
Roots: $0.13653,\ -0.39097.$

Case 3.5. $-2 < b_1 < -1,\ b_2 > 0:$
$b_1 = -1.8,\ b_2 = 4.5:\ f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{0.2}, 0.6^{7.5} \\ 0.6^{-1.8}, 0.6^{4.5} \end{matrix}; 0.6, 0.6x\right] =$
$0.18677x + 0.11129x^2 + 1.$
Roots: $-0.83906 + 2.8777i,\ -0.83906 - 2.8777i.$

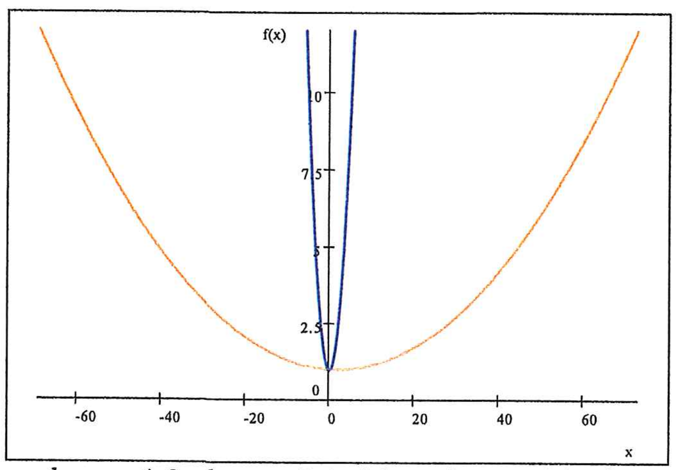
Fig. 10. Case 3.1-Case 3.5: $f(x)$ for different values of b_1, b_2



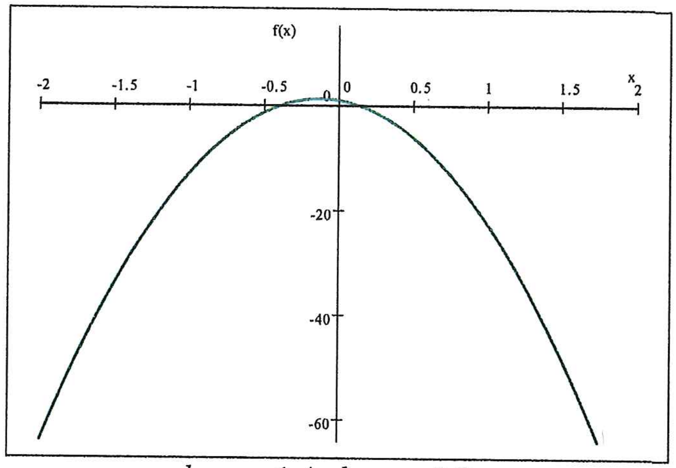
$b_1 = -1.1,\ b_2 = -7.6$



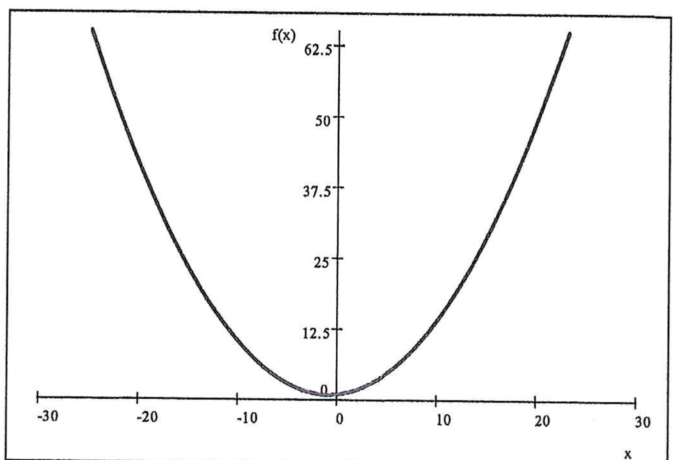
$b_1 = -1.5,\ b_2 = -3.6$



sienna $b_1 = -1.9,\ b_2 = -2.4$ blue $b_1 = -1.7,\ b_2 = -1.4$



$b_1 = -1.4, \ b_2 = -0.2$



$b_1 = -1.8, \ b_2 = 4.5$

Case 4.1. $-1 < b_1 < 0, \ b_2 < -4:$

$b_1 = -0.5, \ b_2 = -4.6: \ f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{1.5}, 0.6^{-1.6} \\ 0.6^{-0.5}, 0.6^{-4.6} \end{matrix} ; 0.6, 0.6x \right] =$
$0.65397x - 8.8658 \times 10^{-2}x^2 + 1.$

Roots: 8.6764, -1.3.

Case 4.2. $-1 < b_1 < 0,\ -4 < b_2 < -3:$

$$b_1 = -0.8,\ b_2 = -3.4: \quad f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{1.2}, 0.6^{-0.4} \\ 0.6^{-0.8}, 0.6^{-3.4} \end{matrix} ; 0.6, 0.6x\right] =$$

$0.11729x + 5.586\,2 \times 10^{-2}x^2 + 1.$

Roots: $-1.0498 + 4.0987i,\ -1.0498 - 4.0987i.$

Case 4.3. $-1 < b_1 < 0,\ -3 < b_2 < -1:$

$$b_1 = -0.4,\ b_2 = -2.0: \quad f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{1.6}, 0.6^{1.0} \\ 0.6^{-0.4}, 0.6^{-2.0} \end{matrix} ; 0.6, 0.6x\right] =$$

$1 - 2.4690x^2 - 1.4778x.$

Roots: $0.40399,\ -1.0026.$

$$b_1 = -0.9,\ b_2 = -1.3: \quad f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{1.1}, 0.6^{1.7} \\ 0.6^{-0.9}, 0.6^{-1.3} \end{matrix} ; 0.6, 0.6x\right] =$$

$1 - 45.109x^2 - 1.2092x.$

Roots: $0.13609,\ -0.16290.$

Case 4.4. $-1 < b_1 < 0,\ -1 < b_2 < 0:$

$$b_1 = -0.1,\ b_2 = -0.6: \quad f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{1.9}, 0.6^{2.4} \\ 0.6^{-0.1}, 0.6^{-0.6} \end{matrix} ; 0.6, 0.6x\right] =$$

$363.70x^2 - 62.258x + 1.$

Roots: $0.15324,\ 1.7943 \times 10^{-2}.$

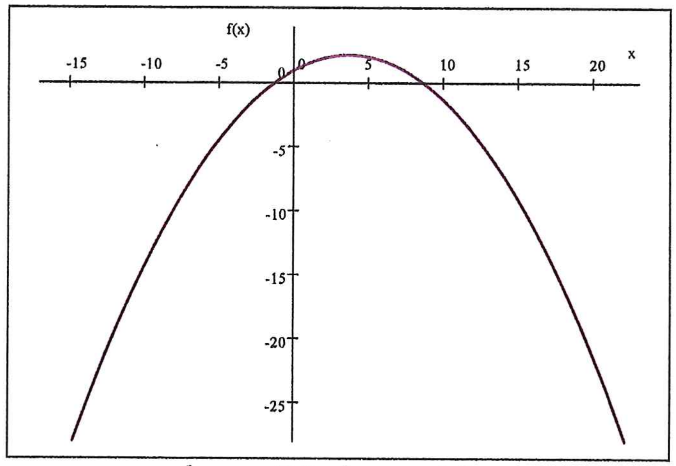
Case 4.5. $-1 < b_1 < 0,\ b_2 > 0:$

$$b_1 = -0.3,\ b_2 = 6.7: \quad f(x) = {}_3\phi_2\left[\begin{matrix} 0.6^{-2}, 0.6^{1.7}, 0.6^{9.7} \\ 0.6^{-0.3}, 0.6^{6.7} \end{matrix} ; 0.6, 0.6x\right] =$$

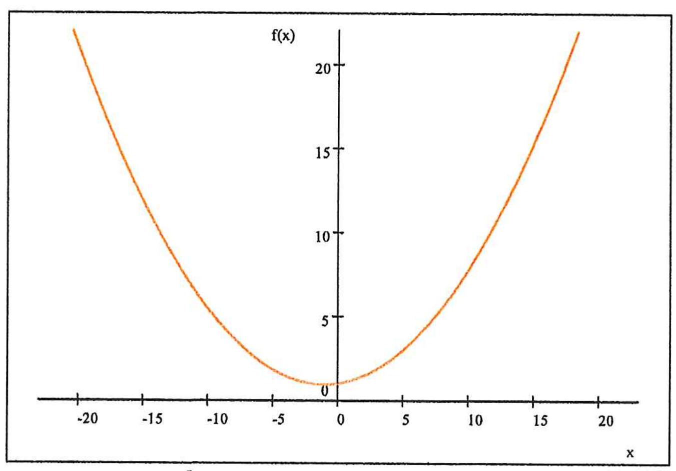
$9.5922x - 15.155x^2 + 1.$

Roots: $0.72409,\ -0.09113.$

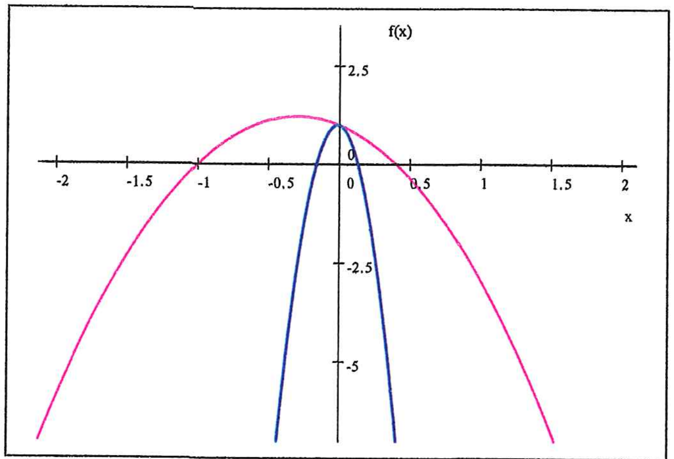
Fig. 11. Case 4.1-Case 4.5: $f(x)$ for different values of b_1, b_2



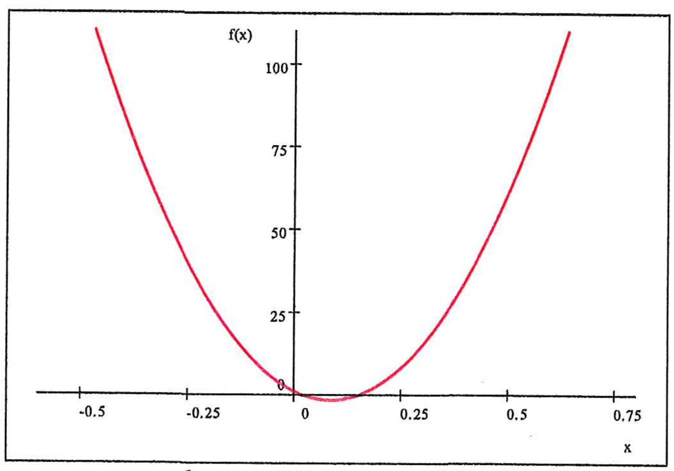
$b_1 = -0.5,\ b_2 = -4.6$



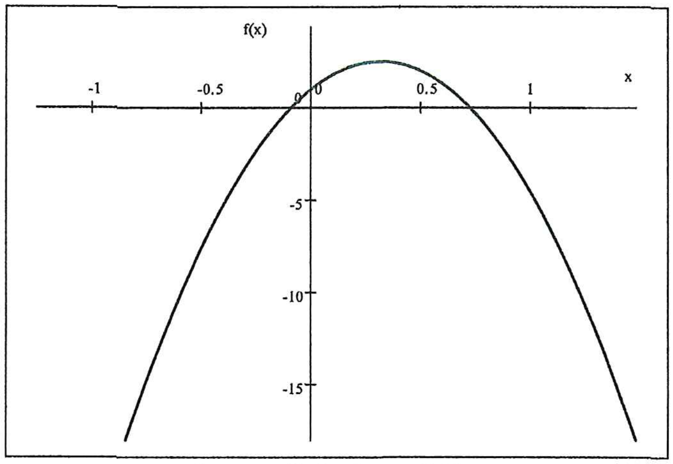
$b_1 = -0.8,\ b_2 = -3.4$



magenta $b_1 = -0.4,\ b_2 = -2.0$ blue $b_1 = -0.9,\ b_2 = -1.3$



$b_1 = -0.1,\ b_2 = -0.6$



$b_1 = -0.3,\ b_2 = 6.7$

Case 5.1. $b_1 > 0,\ b_2 < -4$:
$b_1 = 3.5,\ b_2 = -5.4 :\ f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{5.5}, 0.6^{-2.4} \\ 0.6^{3.5}, 0.6^{-5.4} \end{matrix} ; 0.6, 0.6x \right] =$
$4.0517 \times 10^{-2} x^2 - 0.490\,38x + 1.$
Roots: 9.5069, 2.5961.

Case 5.2. $b_1 > 0,\ -4 < b_2 < -3$:
$b_1 = 6.9,\ b_2 = -3.8 :\ f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{8.9}, 0.6^{-0.8} \\ 0.6^{6.9}, 0.6^{-3.8} \end{matrix} ; 0.6, 0.6x \right] =$
$1 - 4.4406 \times 10^{-3} x^2 - 0.22999x.$
Roots: 4.0338, −55.826.

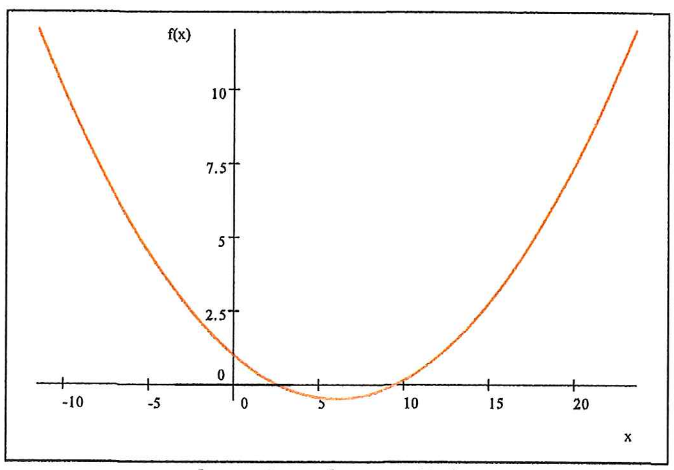
Case 5.3. $b_1 > 0,\ -3 < b_2 < -1$:
$b_1 = 5.2,\ b_2 = -2.9 :\ f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-2}, 0.6^{7.2}, 0.6^{0.1} \\ 0.6^{5.2}, 0.6^{-2.9} \end{matrix} ; 0.6, 0.6x \right] =$
$4.0957 \times 10^{-2} x + 6.9011 \times 10^{-3} x^2 + 1.$
Roots: $-2.9675 + 11.666i,\ -2.9675 - 11.666i.$

$b_1 = 4.1,\ b_2 = -1.6:\ f(x) = {}_3\phi_2\left[\begin{matrix}0.6^{-2}, 0.6^{6.1}, 0.6^{1.4}\\ 0.6^{4.1}, 0.6^{-1.6}\end{matrix};0.6, 0.6x\right] =$
$1.1743x + 1.5196x^2 + 1.$
Roots: $-0.38638 + 0.71328i,\ -0.38638 - 0.71328i.$

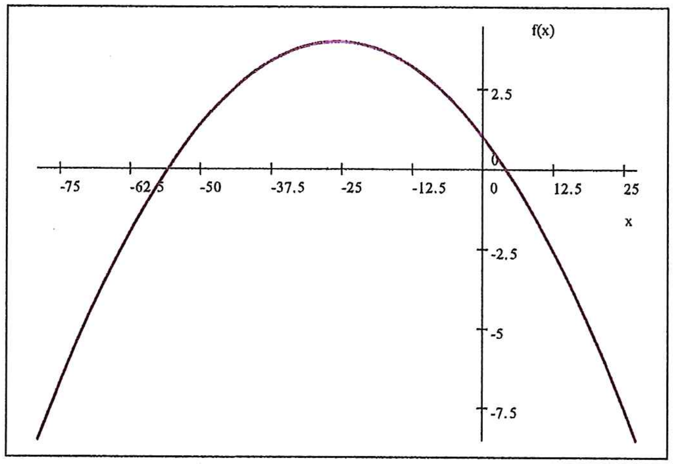
Case 5.4. $b_1 > 0,\ -1 < b_2 < 0:$
$b_1 = 1.2,\ b_2 = -0.7:\ f(x) = {}_3\phi_2\left[\begin{matrix}0.6^{-2}, 0.6^{3.2}, 0.6^{2.3}\\ 0.6^{1.2}, 0.6^{-0.7}\end{matrix};0.6, 0.6x\right] =$
$7.5313x - 35.308x^2 + 1.$
Roots: $0.30589,\ -9.2588 \times 10^{-2}.$

Case 5.5. $b_1 > 0,\ b_2 > 0:$
$b_1 = 2.7,\ b_2 = 6.3:\ f(x) = {}_3\phi_2\left[\begin{matrix}0.6^{-2}, 0.6^{4.7}, 0.6^{9.3}\\ 0.6^{2.7}, 0.6^{6.3}\end{matrix};0.6, 0.6x\right] =$
$2.3750x^2 - 3.3469x + 1.$
Roots: $0.97925,\ 0.42998.$

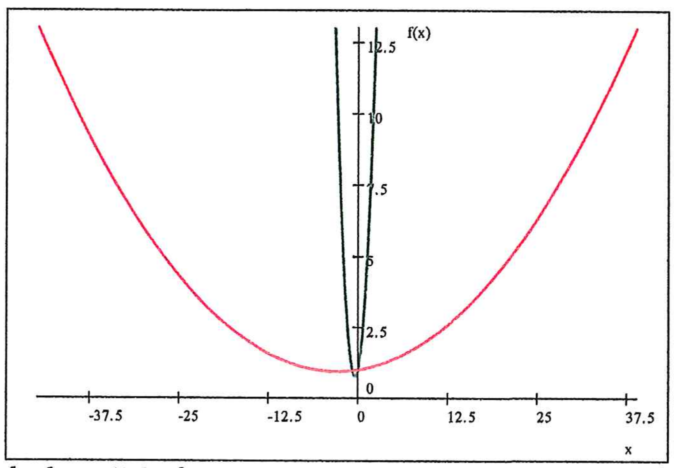
Fig. 12. Case 5.1-Case 5.5: $f(x)$ for different values of b_1, b_2



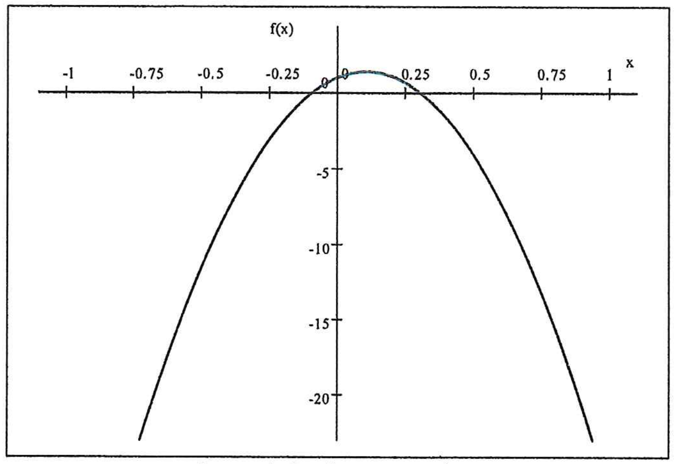
$b_1 = 3.5,\ b_2 = -5.4$



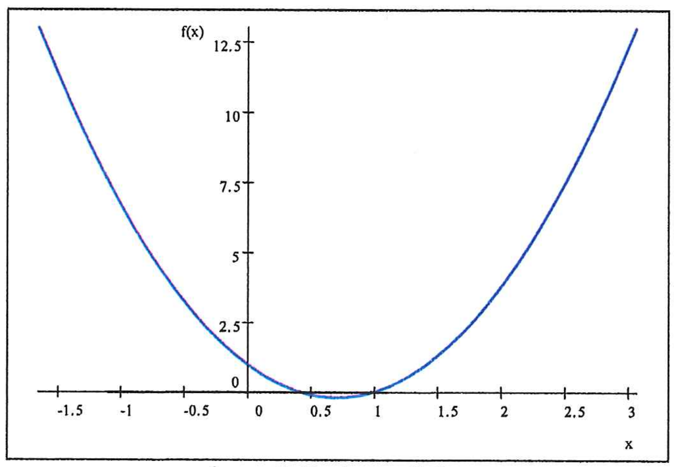
$b_1 = 6.9,\ b_2 = -3.8$



red $b_1 = 5.2,\ b_2 = -2.9$ green $b_1 = 4.1,\ b_2 = -1.6$



$b_1 = 1.2,\ b_2 = -0.7$



$b_1 = 2.7,\ b_2 = 6.3$

The previous results are compiled in the following Table.

Table 5. Characteristics of $f(x)$ for different values of b_1, b_2

Case	Polynomial	Roots	Set of Positivity
1.1 and 5.1 1.5 and 5.5	$a_2x^2 - a_1x + 1$	2 prr: r_1, r_2	$(-\infty; r_1) \cup (r_2; +\infty)$.
1.2 and 5.2	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
1.3 and 5.3	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
1.4 and 5.4	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.1 and 2.5	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.2	$a_2x^2 - a_1x + 1$	2 ir	$(-\infty; +\infty)$.
2.3	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
2.4	$a_1x + a_2x^2 + 1$	2 nrr: r_1, r_2 2 ir	$(-\infty; r_1) \cup (r_2; +\infty)$. $(-\infty; +\infty)$.
3.1 and 3.5	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
3.2	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
3.3	$a_2x^2 - a_1x + 1$	2 ir	$(-\infty; +\infty)$.
3.4	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.1 and 4.5	$a_1x - a_2x^2 + 1$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.2	$a_1x + a_2x^2 + 1$	2 ir	$(-\infty; +\infty)$.
4.3	$1 - a_2x^2 - a_1x$	1 nrr: r_1 1 prr: r_2	$(r_1; r_2)$.
4.4	$a_2x^2 - a_1x + 1$	2 prr: r_1, r_2	$(-\infty; r_1) \cup (r_2; +\infty)$.

3 Effect of the parameters $m_j,\ j = 1, 2, \ldots, r$ on the function ${}_{r+1}\phi_r\left[\cdot\ ; q, q^n x\right]$

The parameters $m_j,\ j = 1, 2, \ldots, r$ influence in the sign and the value of the factors of $(q^{b_j+m_j}; q)_h,\ h = 1, 2, \ldots, M$.

Observe that if $b_j > 0$ then $b_j + m_j > 0$ and according to (6) all the factors of $(q^{b_j+m_j}; q)_h$ are positive.

Now we give two examples to show the effect of $m_j,\ j = 1, 2, \ldots, r$ on the factors of $(q^{b_j+m_j}; q)_h$.

1) Let be $b_j = -6.2, m_j = 5, M = 4$.
From (1) for $b_j + m_j = -1.2$ and $M = 4$ we obtain

$$(q^{-1.2}; q)_h = (1 - q^{-1.2})(1 - q^{-0.2}) \cdots (1 - q^{-2.2+h}),\ h = 1, 2, 3, 4.$$

Table 6. Factors and sign of $(q^{-1.2}; q)_h$ for different values of h

h	$(q^{-1.2}; q)_h$	Sign of $(q^{-1.2}; q)_h$
1	$(1 - q^{-1.2})$	$-$
2	$(1 - q^{-1.2})(1 - q^{-0.2})$	$+$
3	$(1 - q^{-1.2})(1 - q^{-0.2})(1 - q^{0.8})$	$+$
4	$(1 - q^{-1.2})(1 - q^{-0.2})(1 - q^{0.8})(1 - q^{1.8})$	$+$

Therefore, according to (7) and (6), all factors of $(q^{-1.2}; q)_h,\ h = 1, 2,$ are negative, whereas $(q^{-1.2}; q)_h,\ h = 3, 4,$ have 2 negative factors and $h - 2$ positive factors.

2) Let be $b_j = -6.2, m_j = 2, M = 4$.
From (1) for $b_j + m_j = -4.2$ and $M = 4$ we have
$(q^{-4.2}; q)_h = (1 - q^{-4.2})(1 - q^{-3.2}) \cdots (1 - q^{-5.2+h}),\ h = 1, 2, 3, 4.$

Table 7. Factors and sign of $(q^{-4.2};q)_h$ for different values of h

h	$(q^{-4.2};q)_h$	Sign of $(q^{-4.2};q)_h$
1	$(1-q^{-4.2})$	$-$
2	$(1-q^{-4.2})(1-q^{-3.2})$	$+$
3	$(1-q^{-4.2})(1-q^{-3.2})(1-q^{-2.2})$	$-$
4	$(1-q^{-4.2})(1-q^{-3.2})(1-q^{-2.2})(1-q^{-1.2})$	$+$

In this case all factors of $(q^{-4.2};q)_h$, $h = 1,2,3,4$, are negative, according to (7).

3.1 Example: Let be $f(x) = {}_2\phi_1\left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix}; q, q^n x\right]$, $q = 0.5,\ M = 3,\ n = 1.$

From (5) we have

$$f(x) = {}_2\phi_1\left[\begin{matrix} 0.5^{-3}, 0.5^{b+m} \\ 0.5^b \end{matrix}; 0.5, 0.5x\right] = 1 + \frac{(1-0.5^{-3})(1-0.5^{b+m})}{(1-0.5)(1-0.5^b)} \times$$

$$(0.5x) + \frac{(1-0.5^{-3})(1-0.5^{-2})}{(1-0.5)(1-0.5^2)} \frac{(1-0.5^{b+m})(1-0.5^{b+m+1})}{(1-0.5^b)(1-0.5^{b+1})} (0.5x)^2 +$$

$$\frac{(1-0.5^{-3})(1-0.5^{-2})(1-0.5^{-1})}{(1-0.5)(1-0.5^2)(1-0.5^3)} \times$$

$$\frac{(1-0.5^{b+m})(1-0.5^{b+m+1})(1-0.5^{b+m+2})}{(1-0.5^b)(1-0.5^{b+1})(1-0.5^{b+2})} (0.5x)^3 . \qquad (8)$$

3.1.1 Let be $b = 0.1$.

From (8) for different values of m we get,

$m = 1:\ f(x) = {}_2\phi_1\left[\begin{matrix} 0.5^{-3}, 0.5^{1.1} \\ 0.5^{0.1} \end{matrix}; 0.5, 0.5x\right] = 160.29x^2 - 55.765x - 105.53x^3 + 1.$

Roots: 0.9999, 0.50005, 1.8952×10^{-2}.

$m = 2: \; f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{2.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 265.42x^2 - 80.147x - 186.28x^3 + 1.$

Roots: 0.99994, 0.41187, 1.3035×10^{-2}.

$m = 3: \; f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{3.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 325.98x^2 - 92.338x - 235.86x^3 + 1.$

Roots: 0.99159, 0.37923, 1.1275×10^{-2}.

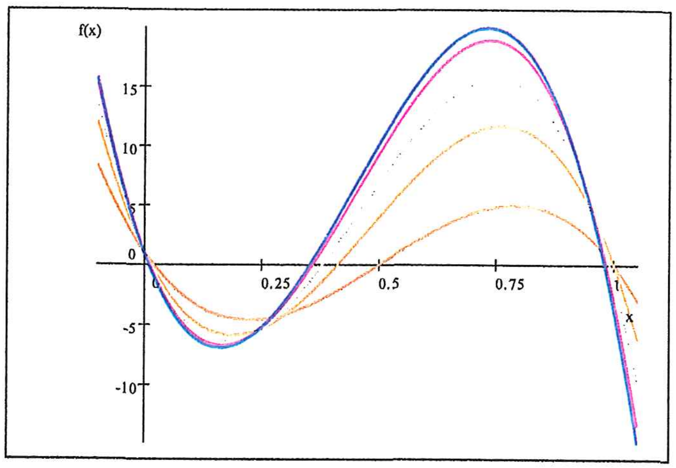


Fig. 13. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.1+m} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right]$ for different values of m

sienna $m = 1$ orange $m = 2$ gray $m = 3$ magenta $m = 5$ blue $m = 10$

$m = 5: \; f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{5.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 374.9x^2 - 101.48x - 277.37x^3 + 1.$

Roots: 0.98326, 0.35813, 1.0238×10^{-2}.

$m = 10: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{10.1} \\ 0.5^{0.1} \end{matrix} ; 0.5, 0.5x \right] = 391.34x^2 - 104.43x - 291.59x^3 + 1.$

Roots: 0.98033, 0.35181, 9.9436×10^{-3}.

Analogously we obtain the following results:

3.1.2 Let be $b = -1.7$

$m = 1: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-0.7} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 2.1125x^3 - 1.1687x^2 - 1.9438x + 1.$

Roots: 1.0, 0.5, -0.94675.

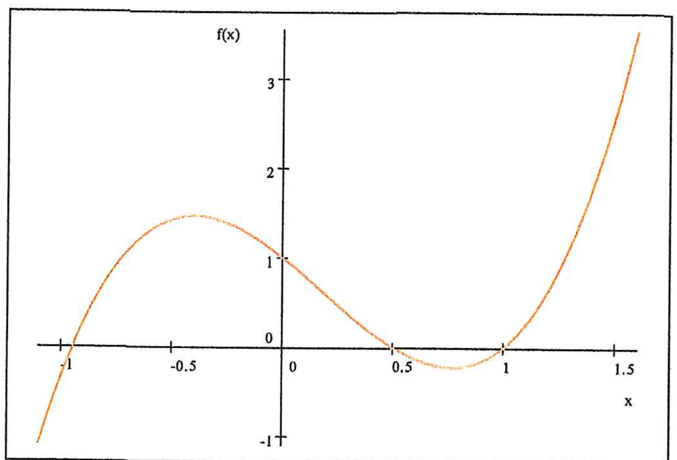


Fig. 14. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-0.7} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right]$

$m = 2: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 0.58436x + 1.1114x^2 - 2.6958x^3 + 1.$

Roots: 1.0, $-0.29386 + 0.53348i$, $-0.29386 - 0.53348i$.

$m = 3: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{1.3} \\ 0.5^{-1.7} \end{matrix} ; 0.5, 0.5x \right] = 1.8484x + 4.7176x^2 - 12.901x^3 + 1.$

Roots: 0.71659, $-0.17545+0.27819i$, $-0.17545-0.27819i$.

$m=4: \ f(x)={}_2\phi_1\left[\begin{array}{c}0.5^{-3},0.5^{2.3}\\ 0.5^{-1.7}\end{array};0.5,0.5x\right]=2.4805x+7.1372x^2-20.620x^3+1.$

Roots: 0.64755, $-0.15071-0.22843i$, $-0.15071+0.22843i$.

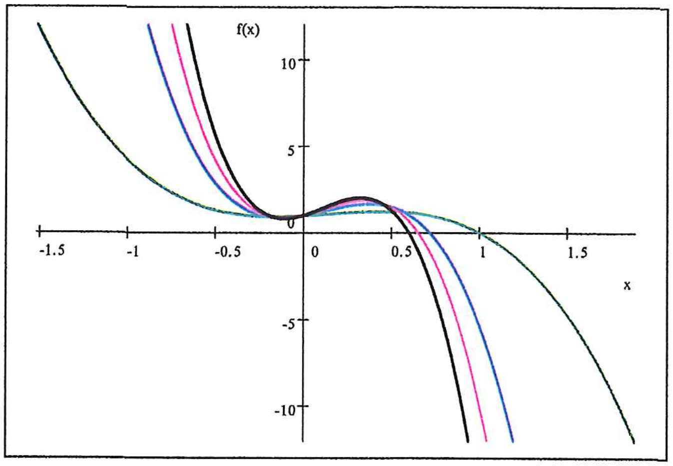


Fig. 15. $f(x)={}_2\phi_1\left[\begin{array}{c}0.5^{-3},0.5^{-1.7+m}\\ 0.5^{-1.7}\end{array};0.5,0.5x\right]$ for different values of m

green $m=2$ blue $m=3$ magenta $m=4$ black $m=8$

$m=8: \ f(x)={}_2\phi_1\left[\begin{array}{c}0.5^{-3},0.5^{6.3}\\ 0.5^{-1.7}\end{array};0.5,0.5x\right]=3.0730x+9.7789x^2-29.669x^3+1.$

Roots: 0.59742, $-0.13391-0.19618i$, $-0.13391+0.19618i$.

3.1.3 Let be $b=-4.5$:

$m=1: \ f(x)={}_2\phi_1\left[\begin{array}{c}0.5^{-3},0.5^{-3.5}\\ 0.5^{-4.5}\end{array};0.5,0.5x\right]=3.0145x^2-3.3382x-0.676\,34x^3+1.$

Roots: 2.9571, 1.0, 0.5.

$m=2: \ f(x) = {}_2\phi_1\left[\begin{array}{c}0.5^{-3}, 0.5^{-2.5}\\ 0.5^{-4.5}\end{array}; 0.5, 0.5x\right] = 0.53442x^2 - 1.5073x - 2.7163\times10^{-2}x^3 + 1.$

Roots: 16.435, 2.2401, 1.0.

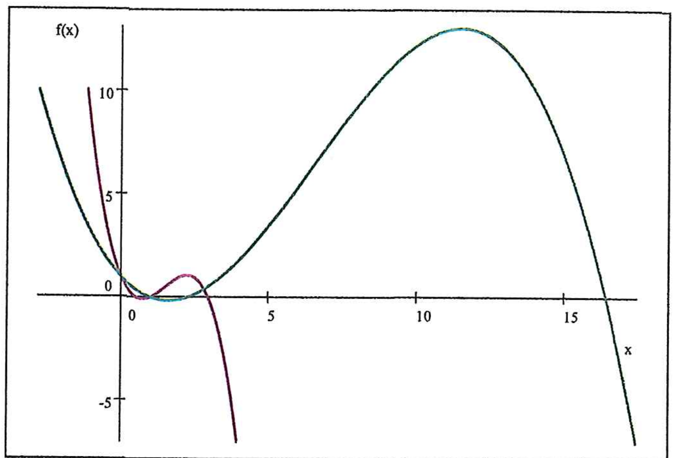


Fig. 16. $f(x) = {}_2\phi_1\left[\begin{array}{c}0.5^{-3}, 0.5^{-4.5+m}\\ 0.5^{-4.5}\end{array}; 0.5, 0.5x\right]$ for different values of m

purple $m=1$ green $m=2$

$m=3: \ f(x) = {}_2\phi_1\left[\begin{array}{c}0.5^{-3}, 0.5^{-1.5}\\ 0.5^{-4.5}\end{array}; 0.5, 0.5x\right] = 4.7535\times10^{-2}x^2 - 0.59179x + 1.7084\times10^{-3}x^3 + 1.$

Roots: 7.6049, 2.0535, -37.482.

$m=4: \ f(x) = {}_2\phi_1\left[\begin{array}{c}0.5^{-3}, 0.5^{-0.5}\\ 0.5^{-4.5}\end{array}; 0.5, 0.5x\right] = 1 - 7.6145\times10^{-3}x^2 - 6.0401\times10^{-4}x^3 - 0.13407x.$

Roots: 5.2457, $-8.9262+15.36i$, $-8.9262-15.36i$.

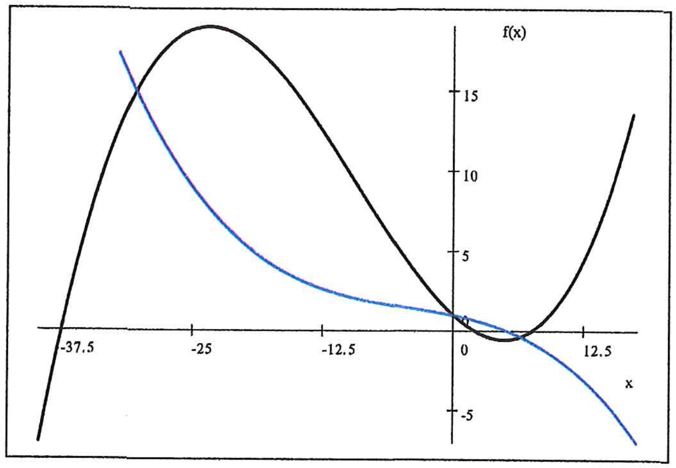


Fig. 17. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-4.5+m} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right]$ for different values of m

black $m = 3$ blue $m = 4$

$m = 5$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{0.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 9.4799 \times 10^{-2}x + 1.1884 \times 10^{-2}x^2 + 1.2004 \times 10^{-3}x^3 + 1.$

Roots: -10.18, $0.14037 - 9.0448i$, $0.14037 + 9.0448i$.

$m = 7$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{2.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 0.26645x + 4.7102 \times 10^{-2}x^2 + 5.5243 \times 10^{-3}x^3 + 1.$

Roots: -5.6569, $-1.4347 + 5.4719i$, $-1.4347 - 5.4719i$.

$m = 12$: $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{7.5} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right] = 0.32188x + 6.2245 \times 10^{-2}x^2 + 7.6273 \times 10^{-3}x^3 + 1.$

Roots: -4.9753, $-1.5927 - 4.8801i$, $-1.5927 + 4.8801i$.

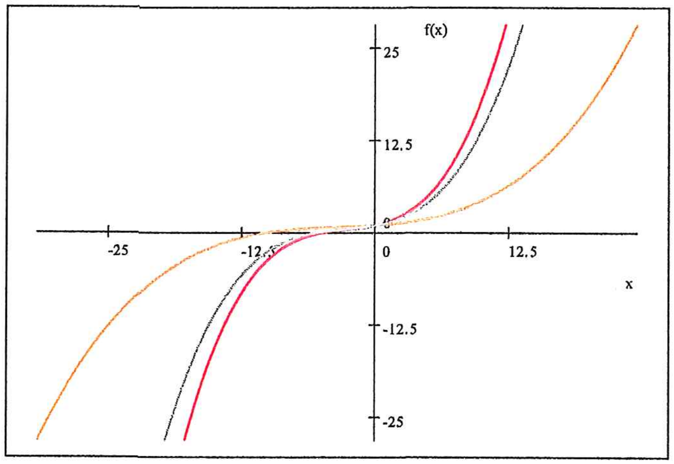


Fig. 18. $f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{-4.5+m} \\ 0.5^{-4.5} \end{matrix} ; 0.5, 0.5x \right]$ for different values of m

sienna $m = 5$ gray $m = 7$ red $m = 12$

Table 8. Characteristics of $f(x)$ for different values of m with b fixed.

b	m	Polynomial	Roots	Set of Positivity
0.1	$1, 2, 3, \cdots$	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
-1.7	1	$a_3x^3 - a_2x^2 - a_1x + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
	$2, 3, 4, \cdots$	$a_1x + a_2x^2 - a_3x^3 + 1$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
-4.5	$1, 2$	$a_2x^2 - a_1x - a_3x^3 + 1$	3 prr: r_1, r_2, r_3	$(-\infty; r_1) \cup (r_2; r_3)$.
	3	$a_2x^2 - a_1x + a_3x^3 + 1$	1 nrr: r_1 2 prr: r_2, r_3	$(r_1; r_2) \cup (r_3; +\infty)$.
	4	$1 - a_2x^2 - a_3x^3 - a_1x$	1 prr: r_1 2 ir	$(-\infty; r_1)$.
	$5, 6, 7, \cdots$	$a_1x + a_2x^2 + a_3x^3 + 1$	1 nrr: r_1 2 ir	$(r_1; +\infty)$.

3.2 Example: $f(x) = {}_3\phi_2\left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x\right]$, $q = 0.3,\ M = 4,\ b_1 = -2.9,\ b_2 = -6.4,\ m_2 = 3,\ n = 4.$

From (5) we get

$$f(x) = {}_3\phi_2\left[\begin{matrix} 0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x\right] =$$

$$1 + \frac{(1-0.3^{-4})(1-0.3^{-2.9+m_1})(1-0.3^{-3.4})}{(1-0.3)(1-0.3^{-2.9})(1-0.3^{-6.4})}(0.3^4 x) +$$

$$\frac{(1-0.3^{-4})(1-0.3^{-3})(1-0.3^{-2.9+m_1})(1-0.3^{-1.9+m_1})}{(1-0.3)(1-0.3^2)(1-0.3^{-2.9})(1-0.3^{-1.9})} \times$$

$$\frac{(1-0.3^{-3.4})(1-0.3^{-2.4})}{(1-0.3^{-6.4})(1-0.3^{-5.4})}(0.3^4 x)^2 +$$

$$\frac{(1-0.3^{-4})(1-0.3^{-3})(1-0.3^{-2})}{(1-0.3)(1-0.3^2)(1-0.3^3)} \times$$

$$\frac{(1-0.3^{-2.9+m_1})(1-0.3^{-1.9+m_1})(1-0.3^{-0.9+m_1})}{(1-0.3^{-2.9})(1-0.3^{-1.9})(1-0.3^{-0.9})} \times$$

$$\frac{(1-0.3^{-3.4})(1-0.3^{-2.4})(1-0.3^{-1.4})}{(1-0.3^{-6.4})(1-0.3^{-5.4})(1-0.3^{-4.4})}(0.3^4 x)^3 +$$

$$\frac{(1-0.3^{-4})(1-0.3^{-3})(1-0.3^{-2})(1-0.3^{-1})}{(1-0.3)(1-0.3^2)(1-0.3^3)(1-0.3^4)} \times$$

$$\frac{(1-0.3^{-2.9+m_1})(1-0.3^{-1.9+m_1})(1-0.3^{-0.9+m_1})(1-0.3^{0.1+m_1})}{(1-0.3^{-2.9})(1-0.3^{-1.9})(1-0.3^{-0.9})(1-0.3^{0.1})} \times$$

$$\frac{(1-0.3^{-3.4})(1-0.3^{-2.4})(1-0.3^{-1.4})(1-0.3^{-0.4})}{(1-0.3^{-6.4})(1-0.3^{-5.4})(1-0.3^{-4.4})(1-0.3^{-3.4})}(0.3^4 x)^4. \quad (9)$$

Then from (9) we have:

$m_1 = 1: \; f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-1.9}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.8935 \times 10^{-5} x^2 - 1.0464 \times 10^{-2} x + 2.0441 \times 10^{-9} x^3 - 2.645 \times 10^{-12} x^4 + 1.$
Roots: 2826.9, 417.47, 123.45, -2595.0.

$m_1 = 2: \; f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-0.9}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 2.75 \times 10^{-13} x^4 - 2.4267 \times 10^{-7} x^2 - 1.6952 \times 10^{-10} x^3 - 2.3116 \times 10^{-3} x + 1.$
Roots: 412.99, 2295.5, $-1046.0 - 1655.8i$, $-1046.0 + 1655.8i$.

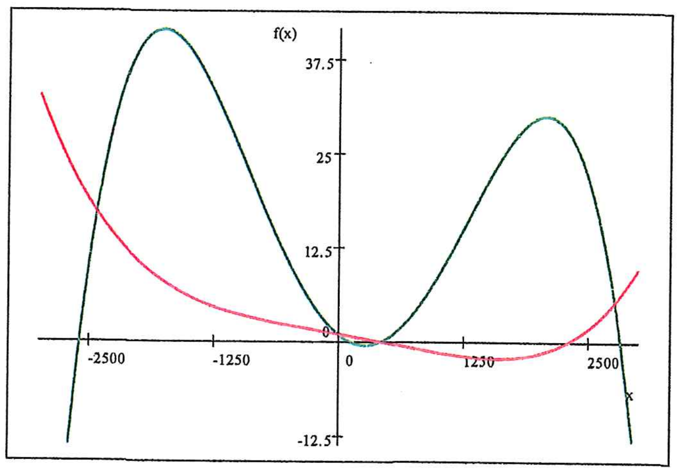


Fig. 19. $f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right]$ for different values of m_1
green $m_1 = 1$ red $m_1 = 2$

$m_1 = 3: \; f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{0.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 1.341 \times 10^{-4} x + 9.1104 \times 10^{-8} x^2 + 7.9784 \times 10^{-11} x^3 - 1.3728 \times 10^{-13} x^4 + 1.$
Roots: $36.853 + 1539.4i$, $36.853 - 1539.4i$, 2024.8, -1517.3.

$m_1 = 4: \; f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{1.1}, 0.3^{-3.4} \\ 0.3^{-2.9}, 0.3^{-6.4} \end{matrix} ; 0.3, 0.3^4 x \right] = 8.678 \times 10^{-4} x + 7.3908 \times 10^{-7} x^2 + 6.8653 \times 10^{-10} x^3 - 1.2016 \times 10^{-12} x^4 + 1.$

Roots: -780.68, 1522.6, $-85.288+832.38i$, $-85.288-832.38i$.

$m_1 = 5 : f(x) = {}_3\phi_2\left[\begin{matrix}0.3^{-4}, 0.3^{2.1}, 0.3^{-3.4}\\ 0.3^{-2.9}, 0.3^{-6.4}\end{matrix}; 0.3, 0.3^4 x\right] = 1.0879 \times 10^{-3}x + 9.8278\times 10^{-7}x^2 + 9.2858 \times 10^{-10}x^3 - 1.6334 \times 10^{-12}x^4 + 1.$
Roots: 1474.2, -702.8, $-101.45+761.98i$, $-101.45-761.98i$.

$m_1 = 10 : f(x) = {}_3\phi_2\left[\begin{matrix}0.3^{-4}, 0.3^{7.1}, 0.3^{-3.4}\\ 0.3^{-2.9}, 0.3^{-6.4}\end{matrix}; 0.3, 0.3^4 x\right] = 1.182 \times 10^{-3}x + 1.0939\times 10^{-6}x^2 + 1.041 \times 10^{-9}x^3 - 1.8352 \times 10^{-12}x^4 + 1.$
Roots: 1456.5, -674.83, $-107.2-736.82i$, $-107.2+736.82i$.

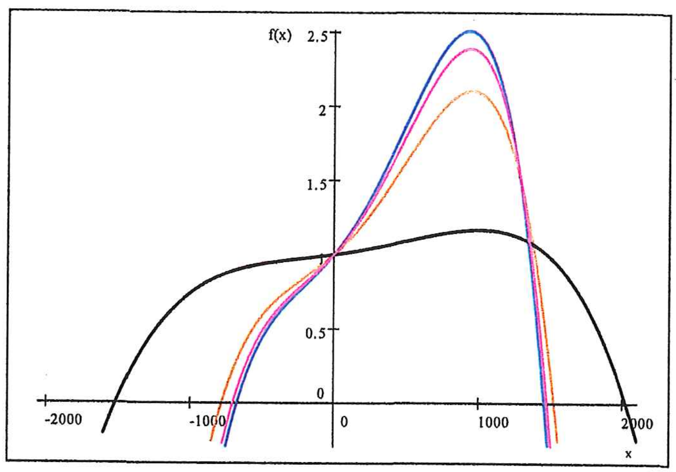


Fig. 20. $f(x) = {}_3\phi_2\left[\begin{matrix}0.3^{-4}, 0.3^{-2.9+m_1}, 0.3^{-3.4}\\ 0.3^{-2.9}, 0.3^{-6.4}\end{matrix}; 0.3, 0.3^4 x\right]$ for different values of m_1

black $m_1 = 3$ sienna $m_1 = 4$ magenta $m_1 = 5$ blue $m_1 = 10$

Table 9. Characteristics of $f(x)$ for different values of m_1

m_1	Polynomial	Roots	Set of Positivity
1	$a_2x^2 - a_1x + a_3x^3 - a_4x^4 + 1$	1 nrr: r_1 3 prr: r_2, r_3, r_4	$(r_1; r_2) \cup (r_3; r_4)$.
2	$a_4x^4 - a_2x^2 - a_3x^3 - a_1x + 1$	2 prr: r_1, r_2 2 ir	$(-\infty; r_1) \cup (r_2; +\infty)$.
$3, 4, 5, \cdots$	$a_1x + a_2x^2 + a_3x^3 - a_4x^4 + 1$	1 nrr: r_1 1 prr: r_2, 2 ir	$(r_1; r_2)$.

4 Effect of q on $f(x)$

4.1 Example: $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-M}, q^{b+m} \\ q^b \end{matrix} ; q, q^n x \right]$, $M = 3$, $b = -2.8$, $m = 6$, $n = 3$.

From (5) we have

$q = 0.9:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.9^{-3}, 0.9^{3.2} \\ 0.9^{-2.8} \end{matrix} ; 0.9, 0.9^3 x \right] = 2.2603x + 3.4834x^2 + 4.994x^3 + 1.$

Roots: $-0.54306, -7.7227 \times 10^{-2} - 0.60230i, -7.7227 \times 10^{-2} + 0.60230i.$

$q = 0.8:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.8^{-3}, 0.8^{3.2} \\ 0.8^{-2.8} \end{matrix} ; 0.8, 0.8^3 x \right] = 1.4348x + 1.4125x^2 + 1.3016x^3 + 1.$

Roots: $-0.85087, -0.11717 + 0.94298i, -0.11717 - 0.94298i.$

$q = 0.7:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.7^{-3}, 0.7^{3.2} \\ 0.7^{-2.8} \end{matrix} ; 0.7, 0.7^3 x \right] = 0.86926x + 0.52476x^2 + 0.29992x^3 + 1.$

Roots: $-1.3901, -0.17977 - 1.5382i, -0.17977 + 1.5382i.$

$q = 0.6:\ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.6^{-3}, 0.6^{3.2} \\ 0.6^{-2.8} \end{matrix} ; 0.6, 0.6^3 x \right] = 0.49615x + 0.17431x^2 +$

$5.8970 \times 10^{-2}x^3 + 1.$
Roots: $-2.3971, -0.27938 + 2.645i, -0.27938 - 2.645i.$

$q = 0.5: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.5^{-3}, 0.5^{3.2} \\ 0.5^{-2.8} \end{matrix} ; 0.5, 0.5^3 x \right] = 0.26148x + 4.9805 \times 10^{-2}x^2 + 9.3394 \times 10^{-3}x^3 + 1.$
Roots: $-4.4491, -0.44182 - 4.8858i, -0.44182 + 4.8858i.$

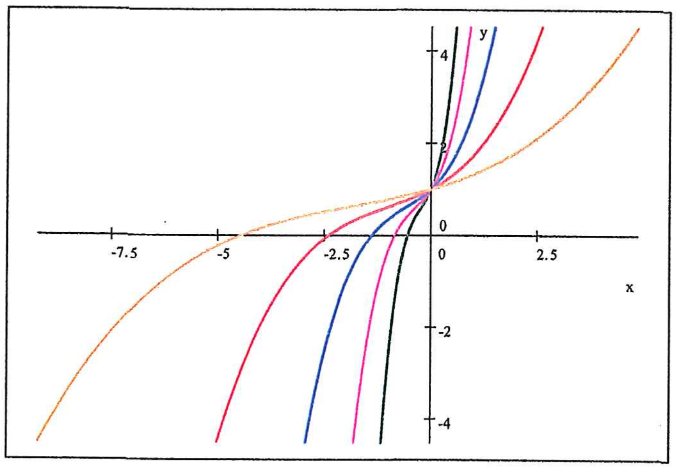


Fig. 21. $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-3}, q^{3.2} \\ q^{-2.8} \end{matrix} ; q, q^3 x \right]$ for different values of q
green $q = 0.9$ magenta $q = 0.8$ blue $q = 0.7$
red $q = 0.6$ sienna $q = 0.5$

$q = 0.4: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.4^{-3}, 0.4^{3.2} \\ 0.4^{-2.8} \end{matrix} ; 0.4, 0.4^3 x \right] = 0.12298x + 1.1454 \times 10^{-2}x^2 + 1.0771 \times 10^{-3}x^3 + 1.$
Roots: $-9.1964, -0.71887 + 10.022i, -0.71887 - 10.022i.$

$q = 0.3: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.3^{-3}, 0.3^{3.2} \\ 0.3^{-2.8} \end{matrix} ; 0.3, 0.3^3 x \right] = 4.8397 \times 10^{-2}x + 1.8655 \times 10^{-3}x^2 + 7.4418 \times 10^{-5}x^3 + 1.$
Roots: $-22.6, -1.2337 + 24.353i, -1.2337 - 24.353i.$

$q = 0.2: \ f(x) = {}_2\phi_1 \left[\begin{matrix} 0.2^{-3}, 0.2^{3.2} \\ 0.2^{-2.8} \end{matrix} ; 0.2, 0.2^3 x \right] = 1.3759 \times 10^{-2} x + 1.6056 \times 10^{-4} x^2 + 1.9735 \times 10^{-6} x^3 + 1.$

Roots: $-76.648, -2.3548 + 81.273i, -2.3548 - 81.273i.$

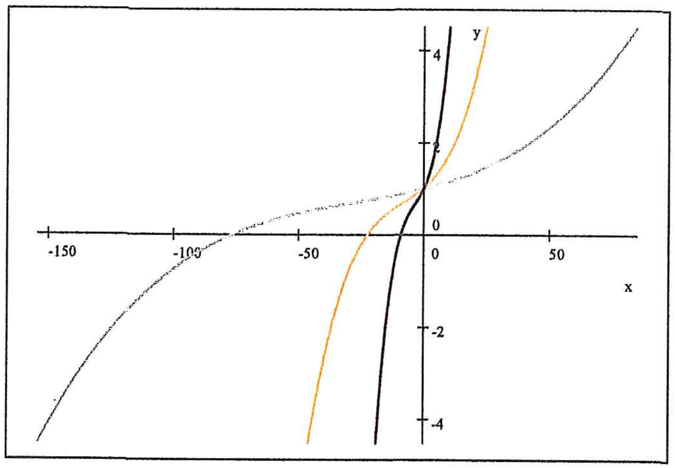


Fig. 22. $f(x) = {}_2\phi_1 \left[\begin{matrix} q^{-3}, q^{3.2} \\ q^{-2.8} \end{matrix} ; q, q^3 x \right]$ for different values of q

purple $q = 0.4$ yellow $q = 0.3$ gray $q = 0.2$

Table 10. Characteristics of $f(x)$ for different values of q

Polynomial: $a_1 x + a_2 x^2 + a_3 x^3 + 1.$
Roots: 1 nrr: r_1, 2 ir.
Set of Positivity: $(r_1; +\infty)$.

From Fig. 21 and Fig. 22 we observe that if q decreases, the real root of $f(x)$ moves further away from the origin.

4.2 Example: $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2} \\ q^{b_1}, q^{b_2} \end{matrix} ; q, q^n x \right]$,

$M = 4, \ b_1 = 0.3, \ m_1 = 1, \ b_2 = 3.5, \ m_2 = 2, \ n = 1.$

From (5) we obtain

$q = 0.8 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.8^{-4}, 0.8^{1.3}, 0.8^{5.5} \\ 0.8^{0.3}, 0.8^{3.5} \end{matrix} ; 0.8, 0.8x \right] = 119.29x^2 - 29.24x - 164.12x^3 + 73.068x^4 + 1.$
Roots: $1.0002, 0.76295, 0.44242, 4.0536 \times 10^{-2}$.

$q = 0.7 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.7^{-4}, 0.7^{1.3}, 0.7^{5.5} \\ 0.7^{0.3}, 0.7^{3.5} \end{matrix} ; 0.7, 0.7x \right] = 145.62x^2 - 32.544x - 216.24x^3 + 102.17x^4 + 1.$
Roots: $0.99968, 0.68944, 0.39104, 3.6316 \times 10^{-2}$.

$q = 0.6 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.6^{-4}, 0.6^{1.3}, 0.6^{5.5} \\ 0.6^{0.3}, 0.6^{3.5} \end{matrix} ; 0.6, 0.6x \right] = 201.64x^2 - 38.83x - 336.63x^3 + 172.83x^4 + 1.$
Roots: $0.99978, 0.59811, 0.31959, 3.0277 \times 10^{-2}$.

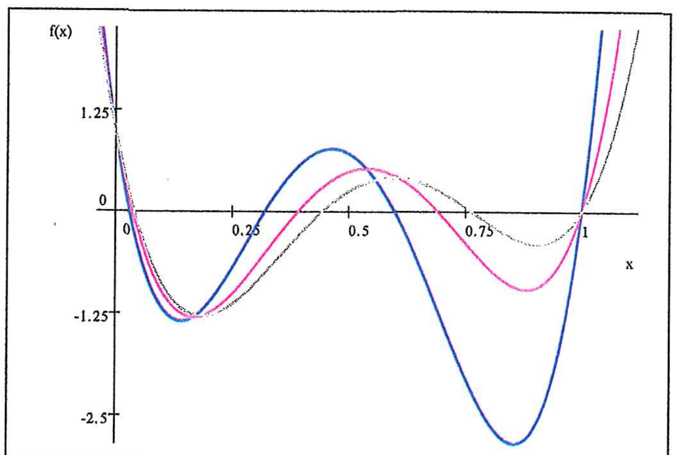


Fig. 23. $f(x) = {}_3\phi_2 \left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx \right]$ for different values of q
gray $q = 0.8$ magenta $q = 0.7$ blue $q = 0.6$

$q = 0.5 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.5^{-4}, 0.5^{1.3}, 0.5^{5.5} \\ 0.5^{0.3}, 0.5^{3.5} \end{matrix} ; 0.5, 0.5x \right] = 329.79x^2 - 50.898x - 648.19x^3 + 368.29x^4 + 1.$
Roots: $1.0001, 0.49969, 0.23736, 2.2892 \times 10^{-2}$.

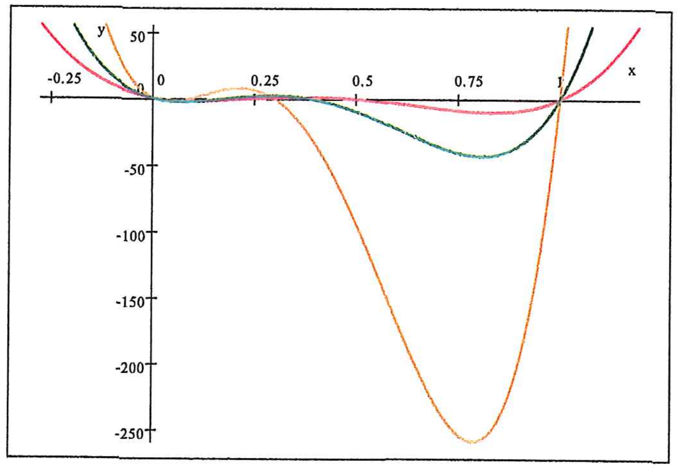


Fig.24. $f(x) = {}_3\phi_2\left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx\right]$ for different values of q
red $q = 0.5$ green $q = 0.4$ sienna $q = 0.3$

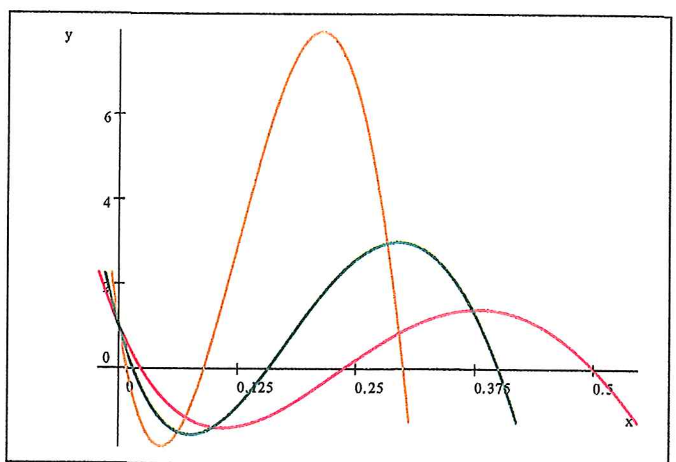


Fig.25. $f(x) = {}_3\phi_2\left[\begin{matrix} q^{-4}, q^{1.3}, q^{5.5} \\ q^{0.3}, q^{3.5} \end{matrix} ; q, qx\right]$ for different values of q with
$-0.0215 < x < 0.546$

$q = 0.4 : f(x) = {}_3\phi_2\left[\begin{matrix} 0.4^{-4}, 0.4^{1.3}, 0.4^{5.5} \\ 0.4^{0.3}, 0.4^{3.5} \end{matrix} ; 0.4, 0.4x\right] = 678.04x^2 -$

$76.102x - 1656.6x^3 + 1053.6x^4 + 1.$
Roots: $1.0001, 0.39992, 0.15719, 1.5097 \times 10^{-2}$.

$q = 0.3 : f(x) = {}_3\phi_2 \left[\begin{matrix} 0.3^{-4}, 0.3^{1.3}, 0.3^{5.5} \\ 0.3^{0.3}, 0.3^{3.5} \end{matrix} ; 0.3, 0.3x \right] = 1962.0x^2 -$
$138.80x - 6410.7x^3 + 4586.5x^4 + 1.$
Roots: $1.0, 0.30001, 8.9614 \times 10^{-2}, 8.1098 \times 10^{-3}$.

Table 11. Characteristics of $f(x)$ for different values of q

Polynomial: $a_2x^2 - a_1x - a_3x^3 + a_4x^4 + 1.$
Roots: 4 prr: $r_1,\ r_2,\ r_3,\ r_4.$
Set of Positivity: $(-\infty; r_1) \cup (r_2; r_3) \cup (r_4; +\infty)$.

From Fig. 23-Fig. 25 we observe that:
i) if q decreases then the roots r_1, r_2, r_3 decrease, while the value of r_4 presents small downs and ups, with $r_1 < r_2 < r_3 < r_4$.
ii) $(f_{\max})_{q=0.3} > (f_{\max})_{q=0.4} > \cdots > (f_{\max})_{q=0.7} > (f_{\max})_{q=0.8}$.
$(f_{\min})_{q=0.3} < (f_{\min})_{q=0.4} < \cdots < (f_{\min})_{q=0.7} < (f_{\min})_{q=0.8}$.

5 Effect of n on the function ${}_{r+1}\phi_r\left[\cdot\ ; q, q^n x\right]$

5.1 Example: $f(x) = {}_4\phi_3 \left[\begin{matrix} q^{-M}, q^{b_1+m_1}, q^{b_2+m_2}, q^{b_3+m_3} \\ q^{b_1}, q^{b_2}, q^{b_3} \end{matrix} ; q, q^n x \right]$, $q = 0.9,\ M = 3,\ b_1 = -0.5,\ m_1 = 3,\ b_2 = 2.9,\ m_2 = 1,\ b_3 = -3.4,\ m_3 = 2.$

Observe that q^n is a factor of scale for x and $0 < q^n < 1$.
We consider the function $g(x)$ defined by

$g(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, x \right] = 7.5144x - 9.983x^2 -$
$4.6776x^3 + 1.$
Roots: $r_3 = 0.68182, r_2 = -0.11613, r_1 = -2.6999.$

Then we can obtain $f(x)$ expanding $g(x)$ in $(0.9)^n$ units, since $(0.9)^n < 1$, that is,
$f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, (0.9)^n x\right] = 7.5144\left((0.9)^n x\right) - 9.983\left((0.9)^n x\right)^2 - 4.6776\left((0.9)^n x\right)^3 + 1.$

So the roots of $f(x)$ are $\frac{r_3}{(0.9)^n}, \frac{r_2}{(0.9)^n}, \frac{r_1}{(0.9)^n}$.

In this way we get the following results:
$n = 1: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9x\right] = 6.7630x - 8.0862x^2 - 3.4100x^3 + 1.$
Roots: $0.75758, -0.12904, -2.9999$.

$n = 2: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^2x\right] = 6.0867x - 6.5499x^2 - 2.4859x^3 + 1.$
Roots: $0.84175, -0.14338, -3.3332$.

$n = 3: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^3x\right] = 5.478x - 5.3054x^2 - 1.8122x^3 + 1.$
Roots: $0.93527, -0.15931, -3.7036$.

$n = 4: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^4x\right] = 4.9302x - 4.2974x^2 - 1.3211x^3 + 1.$
Roots: $1.0392, -0.17701, -4.1151$.

$n = 5: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^5x\right] = 4.4372x - 3.4809x^2 - 0.96308x^3 + 1.$
Roots: $1.1547, -0.19667, -4.5723$.

$n = 8: \; f(x) = {}_4\phi_3\left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^8x\right] = 3.2347x-$

$1.8499x^2 - 0.37312x^3 + 1.$
Roots: $1.5839, -0.26979, -6.272.$

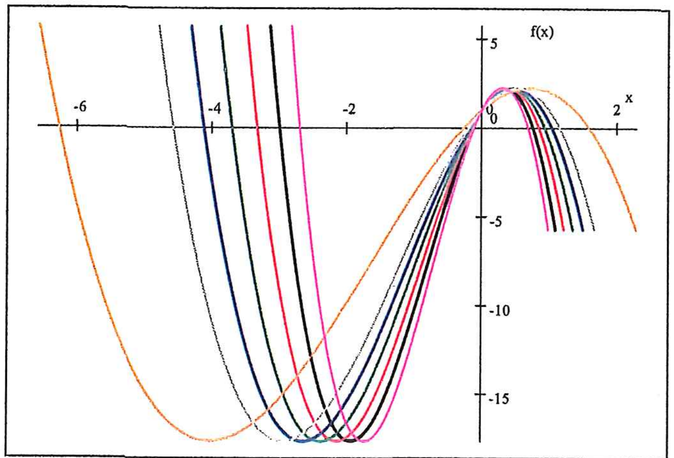


Fig. 26. $f(x) = {}_4\phi_3 \left[\begin{matrix} 0.9^{-3}, 0.9^{2.5}, 0.9^{3.9}, 0.9^{-1.4} \\ 0.9^{-0.5}, 0.9^{2.9}, 0.9^{-3.4} \end{matrix} ; 0.9, 0.9^n x \right]$ for different values of n

black $n = 1$ red $n = 2$ green $n = 3$ blue $n = 4$ gray $n = 5$ sienna $n = 8$
$g(x)$: magenta

Table12. Characteristics of $f(x)$ for different values of n

Polynomial: $a_1 x - a_2 x^2 - a_3 x^3 + 1.$
Roots: 2 nrr: r_1, r_2; 1 prr: r_3.
Set of Positivity: $(-\infty; r_1) \cup (r_2; r_3)$.

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