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THE CAUSAL SPIRALS ON THE SPHERES OF LORENTZ-MINKOWSKI 3-SPACE $\mathbb{R}^3_1$

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Abstract

Spirals are differentiable curves cutting all meridians (or all parallels) of a rotational surface with a constant angle. In this paper, we obtain differential equations of all spirals on the spheres of Lorentz-Minkowki 3-space $\mathbb{R}^3_1$ and define the general parametrizations of these curves which are the solutions of the differential equations.

keywords: Hyperbolic spherical spiral; Lorentzian spherical spirals; lightlike conical spirals; Minkowski 3-Space.

1 Introduction

The research area of the spherical curves, which has characteristic properties on sphere, attracts many authors' attention in differential geometry [4, 6, 10, 18, 20]. For instance, the curves, that are the intersection of the sphere with planes passing through the origin, give the property of shortest path on the sphere and are called geodesics. In this sense, spherical helices are important curves that make a constant angle with a fixed axis [6]. On the other hand, the spherical curves that cut the meridians (or parallels) with a constant angle are called spherical spirals (also known as loxodromes or rhumb lines), which are not geodesic [6, 11, 18, 20]. These special curves are widely used in aviation and navigation. It is well known that the spherical spirals look like straight lines and logarithmic spirals under the mercator projections [18] and the stereographic projections [6], respectively.

Spherical curves has been also studied in Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ by the researchers. For example, Petrović and Šućurović [16] presented some characterizations of the Lorentzian spherical timelike and null curves. They showed that there are no non-geodesic null curves on $S^2_1(r)$. Inoguchi and Lee [8] gave some characterizations of null helices in terms of their associated curves. They examined the geometry of null curves and presented the important applications of null curves. Besides, they proved the existence of two geodesic null lines on $S^2_1(r)$. Liu and Meng [13] gave the representation formulas for spacelike curves on lightlike cones Λ^2 and Λ^3. Babaarslan and Yaylı [2] examined the spacelike loxodromes on rotational surfaces which have spacelike meridians or timelike meridians. It is well-known that the Lorentzian sphere $S^2_1(r)$ and lightlike cone Λ^2 can be parameterized as ruled surfaces. Therefore, the research areas of ruled surfaces in the space $\mathbb{R}^3_1$ attract many researchers. For Example, Alías et al. [1] showed that B-scrolls over null curves are the ruled surfaces in 3-dimensional Lorentz-Minkowski space L^3 with null rulings whose Gauss map G satisfies the condition $\Delta G = \Lambda G$, where Δ denotes the Laplace operator of the surface and Λ is an endomorphism of L^3. Kılıç et al. [9] examined the special null curves on the ruled surfaces and Liu [12] gave some characterizations of ruled surfaces with lightlike ruling in Minkowski 3-space. Choi and Kim [5] gave characterizations of some special helices and self associated curves of a null curve and explicit representations of these curves, characterized general and slant helices in terms of their associated curves.

There are many models of hyperbolic geometry from non-Euclidean geometries. In these models, the first four postulates of Euclid are provided while the fifth postulate is not provided. The hyperboloid model, also known as the Minkowski model or the Lorentz model [15, 17, 19] is a model of 3-dimensional hyperbolic geometry in which points are represented by the points on the positive sheet $H^+(r)$ of Euclidean two-sheeted hyperboloid $H_0^2(r)$, the set of the unit timelike vectors of constant length r in 3-dimensional Minkowski space $\mathbb{R}_1^3$. The restriction of the quadric form induced by Lorentz metric $\langle . \, , . \rangle$ to the tangent plane of $H^+(r)$ is positive definite. So, it gives a Riemannian metric on $H^+(r)$ producing a model of 3-dimensional hyperbolic space H^3.

The meridians of $H^+(r)$ are hyperbolic circles which are intersecting with timelike planes passing through the origin. The parallels are Euclidean circles which are intersecting with planes $z = k$, $(k > 1)$. While the meridians are geodesic, the parallels are not geodesic. Since $H^+(r)$ is a spacelike surface, there must be spacelike spirals on the surface that cuts all meridians under *a constant spacelike angle.* The set of spacelike vectors with length r is called Lorentzian sphere and is denoted by $S_1^2(r)$. Note that Lorentzian sphere $S_1^2(r)$ is de Sitter Space-Time (See, [19]). The meridians of $S_1^2(r)$ are Lorentzian circles which are intersecting with planes including the origin and cover the z-axis; the parallels are Euclidean circles which are intersecting with planes $z = k$, $(k \in \mathbb{R})$. The parallels are not geodesic, except for the equator while the meridians are geodesics. Since $S_1^2(r)$ is a timelike surface, there must be three types of spirals on the surface: These are timelike spiral that cuts all meridians under *a constant hyperbolic angle*, spacelike spiral that cuts all meridians under *a constant timelike angle* and lightlike spiral that cuts all meridians under *a constant lightlike angle.* The meridians of Λ^2 are lightlike lines which are intersecting with lightlike planes passing through the origin. The parallels are Euclidean circles which are intersecting with planes $z = k$, $(k \in \mathbb{R})$. While the meridians are geodesic, the parallels are not geodesic. Since Λ^2 is a lightlike surface, there exists only spacelike spiral on the surface that cuts all parallels under *a constant Lorentzian angle* (*angle with a measure of zero number*).

In this paper, we obtain the differential equations of spirals on the surfaces of hyperboloid model $H^+(r)$, of the Lorentzian sphere $S_1^2(r)$ and of the lightlike cone Λ^2 in the Minkowski 3–space $\mathbb{R}_1^3$, and define the general parametrizations of causal spirals which are the solutions of the differential equations.

2 Basic Concepts

Let $\mathbb{R}_1^3$ be a Minkowski 3-space provided with natural Lorentzian metric given by

$$\langle . \, , . \rangle = dx_1^2 + dx_2^2 - dx_3^2, \tag{2.1}$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{R}_1^3$. An arbitrary vector $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ is said to be spacelike (resp. timelike, lightlike) if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ or $\boldsymbol{v} = 0$ (resp. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ and $\boldsymbol{v} \neq 0$). The norm of a vector $\boldsymbol{v} \in \mathbb{R}_1^3$ is defined by $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. Thus, a spacelike (timelike) vector $\boldsymbol{v}$ is unit if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$ $(\langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1)$.
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in the space $\mathbb{R}_1^3$. The Lorentzian cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} \times \boldsymbol{v} = (u_3 v_2 - u_2 v_3, u_1 v_3 - u_3 v_1, u_1 v_2 - u_2 v_1), \tag{2.2}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. For standard base vectors, we have

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3, \quad \boldsymbol{e}_2 \times \boldsymbol{e}_3 = -\boldsymbol{e}_1, \quad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = -\boldsymbol{e}_2.$$

Let V be a 2-dimensional linear subspace $\mathbb{R}_1^3$. Then, there are three mutually exclutive possibilities for V :

(i) V is said to be spacelike if the restriction $\langle . \, , . \rangle|_V$ of the Lorentzian metric on V is positive definite.

(ii) V is said to be timelike if the restriction $\langle . \, , . \rangle\, |_V$ of the Lorentzian metric on V is Lorentzian, i.e, non-degenerate and of the signature (1). Then, V is a timelike plane.

(iii) V is lightlike (or null) if $\langle .\, , . \rangle |_V$ is degenerate.

The spheres of the space $\mathbb{R}^3_1$ are defined as follows:
The set of all spacelike vectors of radius $r > 0$ with origin-centered is called Lorentzian sphere of radius r and given by

$$S^2_1(r) = \{ \boldsymbol{v} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = r^2 \}.$$

The sphere $S^2_1(r)$ is a Lorentzian 2-manifold of constant sectional curvature $1/r^2$. On the other hand, for $r > 0$, the quadric

$$H^2_0(r) = \{ \boldsymbol{v} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -r^2 \}$$

is called hyperbolic sphere of radius r. The sphere $H^2_0(r)$ is also a Riemannian 2-manifold of constant sectional curvature $-1/r^2$. This quadric has two connected components given by

$$H^+(r) = \{ \boldsymbol{v} \in H^2_0(r) \mid \langle \boldsymbol{v}, \boldsymbol{e}_3 \rangle < 0 \},$$
$$H^-(r) = \{ \boldsymbol{v} \in H^2_0(r) \mid \langle \boldsymbol{v}, \boldsymbol{e}_3 \rangle > 0 \}.$$

Each component is a simply connected Riemannian manifold of constant curvature $-1/r^2$. The set Λ^2 of all lightlike vectors

$$\Lambda^2 = \{ \boldsymbol{v} \in \mathbb{R}^3_1 - \{0\} \mid \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \}$$

is called lightlike cone of $\mathbb{R}^3_1$. This quadric has also two components given by

$$\Lambda^+ = \{ \boldsymbol{v} \in \Lambda^2 \mid \langle \boldsymbol{v}, \boldsymbol{e}_3 \rangle < 0 \},$$
$$\Lambda^- = \{ \boldsymbol{v} \in \Lambda^2 \mid \langle \boldsymbol{v}, \boldsymbol{e}_3 \rangle > 0 \}.$$

The components Λ^+ and Λ^- is called future cone and past cone, respectively. Then a ray in Λ^+ stating at the origin corresponds to a point on boundary of H^3. The set of such rays form the sphere at infinity $S^2_\infty = \partial H^3$.
For an arbitrary point u in Λ^+ Heard [7] defines the horosphere as

$$h_{\boldsymbol{u}} := \{ \boldsymbol{x} \in H^+ \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = -1 \}$$

which inherits an Euclidean structure. Note that $\boldsymbol{u}$ is the position vector field of the point u.

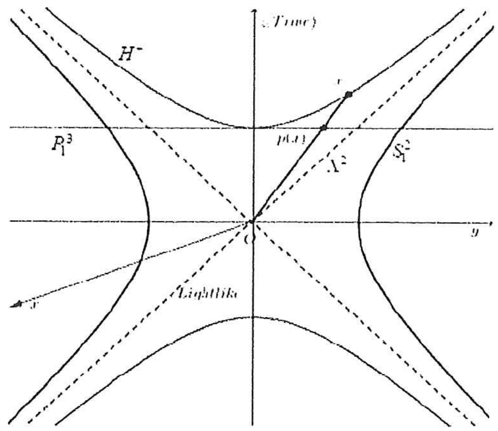

Figure 1: The space $\mathbb{R}_1^3$ equipped with the Lorentzian inner product $\langle . \, , . \rangle$ is 3 dimensional Lorentzian space.

Let denote the radial projection by p. Then the projection is given by

$$p: \{\boldsymbol{x} \in \mathbb{R}_1^3 \mid x_3 \neq 0\} \to P_1^3 = \{\boldsymbol{x} \in \mathbb{R}_1^3 \mid x_3 = 1\},$$

where P_1^3 is affine plane along the rays through the origin. The projection p is a homeomorphism from H^+ onto the 3-dimensional open unit ball B^3 in P_1^3 centered at the origin $(0,0,1)$ of P_1^3, which give the projective model of H^3. The affine plane P_1^3 contains B^3 and its set theoretic boundary ∂B^3 in P_1^3, which is identified with S_∞^2. Then we have $\bar{B}^3 = B^3 \cup \partial B^3$.
Now let define the geodesic plane $\boldsymbol{u}^\perp$ for an arbitrary point u in S_1^2 as

$$\boldsymbol{u}^\perp := \{\boldsymbol{x} \in H^3 \mid <\boldsymbol{x}, \boldsymbol{u}> = 0\}.$$

A point u also defines a half-space in H^3 given by

$$\Pi_u := \{\boldsymbol{x} \in H^3 \mid <\boldsymbol{x}, \boldsymbol{u}> \leq 0\},$$

where $\boldsymbol{u}$ is the position vector field of u.

Definition 2.1 *The signed distance d between horosphere and a plane (respectively, point, horosphere) is the distance by which the horosphere extends past the plane (respectively, point, horosphere). The distance d may be positive, negative, or zero, as shown in (Figure 2).*
(For hyperbolic distances, see [7]).

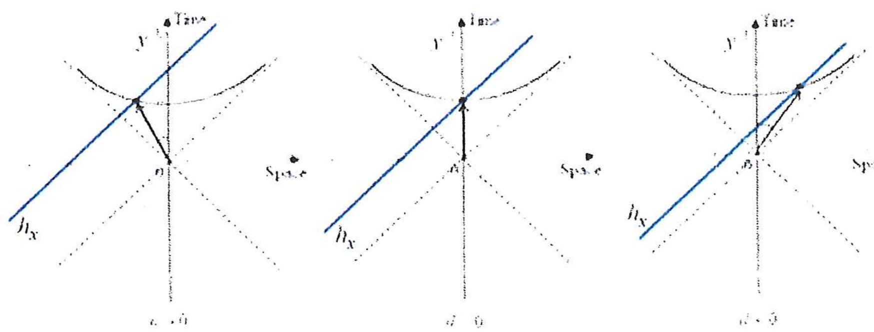

Figure 2: The signed distance from a plane to a horosphere is the distance d by which the horosphere extends to the plane.

3 The Angles in the Minkowski Space $\mathbb{R}^3_1$

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in the space $\mathbb{R}^3_1$. The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined with respect to these causal characters as follows [3, 7, 14, 15, 17, 19]:

(i) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two timelike vectors. Then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \geq \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. In this case, both vectors lie in the same cone and there is a unique real number $\theta \geq 0$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -\|\boldsymbol{u}\|\,\|\boldsymbol{v}\| \cosh\theta. \tag{3.1}$$

This number is called *hyperbolic angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(ii) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two spacelike vectors and they span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\| \cosh\theta. \tag{3.2}$$

This number is called *central angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iii) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two spacelike vectors and they span a spacelike vector subspace. Then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. In this case, there is a unique real number $\theta \geq 0$ such that

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta. \tag{3.3}$$

This number is called *spacelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iv) Let $\boldsymbol{u}$ be a spacelike vector and $\boldsymbol{v}$ be a timelike vector. Then there is a unique real number $\theta \geq 0$ such that

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \sinh\theta. \tag{3.4}$$

This number is called *timelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(v) Two geodesic planes $\boldsymbol{u}^{\perp}$ and $\boldsymbol{v}^{\perp}$ do not intersect in H^2 but intersect in ∂H^2 if and only if

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \cos 0 = 1. \tag{3.5}$$

In this case the *Lorentzian angle* between them is zero number. It must be note that the Lorentzian angles correspond to hyperbolic distances between two points on the hyperboloid model $H^+(r)$.

(vi) Let $\boldsymbol{u} \in \Lambda^+$ and $\boldsymbol{v} \in S_1^2$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}$ and $h_{\boldsymbol{u}}$. This angle satisfies the equality

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = e^{-d}.$$

(vii) Let $\boldsymbol{u} \in \Lambda^+$ and $\boldsymbol{v} \in H^+$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}^{\perp}$ and $h_{\boldsymbol{u}}$. This angle satisfies the equality

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -e^{-d}.$$

(viii) Let $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^+$. The *lightlike angle* between them is signed distance d between $h_{\boldsymbol{u}}$ and $h_{\boldsymbol{v}}$. This angle satisfies the equality

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = 2e^{-d}.$$

(For the geometrical interpretations of angles in Lorentzian 3-space, see [7, 17]).

4 The Spirals on the Hyperboloid Model $H^+(r)$

The hyperbolic sphere with center O and radius r is the surface given by

$$H_0^2(r) = \left\{(x, y, z) \in \mathbb{R}_1^3 \,|\, x^2 + y^2 - z^2 = -r^2\right\}.$$

We observe that the set $H_0^2(r)$ has exactly two connected components. Thus, the hyperboloid model of hyperbolic geometry from non-Euclidean geometries in the Lorentz-Minkowski space is denoted by

$$H^+(r) = \left\{(x, y, z) \in \mathbb{R}_1^3 \,|\, x^2 + y^2 - z^2 = -r^2,\ z > 0\right\}.$$

This is a spacelike surface and called a hyperbolic plane [7, 17, 19]. We shall compute the first fundamental form of $H^+(r)$ at a point of the coordinate neighborhood given by parametrization

$$x(u, v) = (r\,\mathrm{sinh}u\,\mathrm{cos}v, r\,\mathrm{sinh}u\,\mathrm{sin}v, r\,\mathrm{cosh}u) \tag{4.1}$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. First, observe that

$$\begin{cases} \boldsymbol{x}_u(u, v) = (r\,\mathrm{cosh}u\,\mathrm{cos}v, r\,\mathrm{cosh}u\,\mathrm{sin}v, r\,\mathrm{sinh}u) \\ \boldsymbol{x}_v(u, v) = (-r\,\mathrm{sinh}u\,\mathrm{sin}v, r\,\mathrm{sinh}u\,\mathrm{cos}v, 0). \end{cases} \tag{4.2}$$

Hence,

$$E(u, v) = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = r^2,\ F(u, v) = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0,\ G(u, v) = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle = r^2\mathrm{sinh}^2 u. \tag{4.3}$$

If $\boldsymbol{w}$ is a tangent vector to the sphere at the point $x(u, v)$, then $\boldsymbol{w}$ is given in the basis associated to $x(u, v)$ by

$$\boldsymbol{w} = a\boldsymbol{x}_u + b\boldsymbol{x}_v.$$

Thus, the square of the length of $\boldsymbol{w}$ is given by

$$|\boldsymbol{w}|^2 = I(\boldsymbol{w}) = Ea^2 + 2Fab + Gb^2 = a^2r^2 + b^2r^2\mathrm{sinh}^2 u.$$

Let us determine the spirals in this coordinate neighborhood of the hyperboloid model $H^+(r)$ which make a constant angle θ with the meridians ($v = const.$). In this case, we give definitions as follows:

Definition 4.1 *Let $\alpha_\lambda(t)$ be a differentiable curve on the surface of the hyperboloid model $H^+(r)$. The curve $\alpha_\lambda(t)$ is called hyperbolic spherical spiral if $\alpha_\lambda(t)$ cuts all meridians of $H^+(r)$ with a constant spacelike angle.*

Theorem 4.2 *Let θ be constant spacelike angle between the meridians and the spacelike curve $\alpha_\lambda(t)$ and assume that $\lambda = \cot\theta$. Then the parametrization of all spirals on the surface of the hyperboloid model $H^+(r)$ is given by*

$$\alpha_\lambda(t) = \left(\frac{2re^{\lambda t}}{e^{2\lambda t}-1} \cos t, \frac{2re^{\lambda t}}{e^{2\lambda t}-1} \sin t, r\frac{e^{2\lambda t}+1}{e^{2\lambda t}-1} \right). \tag{4.4}$$

Proof. We may assume that the required curve

$$\alpha_\lambda(t) = x(u(t), v(t)) = (r\sinh u(t)\cos v(t), r\sinh u(t)\sin v(t), r\cosh u(t))$$

is the image by x of a curve $(u(t), v(t))$ of the uvˇplane. In this case we have

$$\begin{aligned}\boldsymbol{\alpha}'_\lambda(t) = (&r\,u'(t)\cosh u(t)\cos v(t) - rv'(t)\sinh u(t)\sin v(t),\\ &r\,u'(t)\cosh u(t)\sin v(t) + rv'(t)\sinh u(t)\cos v(t), ru'(t)\sinh u(t)).\end{aligned}$$

At the point $x(u, v)$ where the curve meets the meridians ($v = const.$), we have

$$\cos\theta = \frac{|\langle \boldsymbol{x}_u, \boldsymbol{\alpha}'_\lambda(t)\rangle|}{\|\boldsymbol{x}_u\|\,\|\boldsymbol{\alpha}'_\lambda(t)\|} = \frac{u'}{\sqrt{(u')^2 + (v')^2\sinh^2 u}}, \tag{4.5}$$

since in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ the vector $\boldsymbol{\alpha}'_\lambda(t)$ has coordinates (u', v') and the vector $\boldsymbol{x}_u$ has coordinates $(1, 0)$. By (12) we obtain the differantial equation as

$$(u')^2\tan^2\theta - (v')^2\sinh^2 u = 0$$

or

$$\frac{du}{\sinh u} = \pm\frac{dv}{\tan\theta}. \tag{4.6}$$

Hence, by integration we obtain the differential equation of the hyperbolic spherical spirals as

$$\ln\left(\tanh\frac{u}{2}\right) = \pm(v + c)\cot\theta, \tag{4.7}$$

where the constant of integration c is to be determined by giving one point $x(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the differential equation (14) gives

$$u(t) = 2\tanh^{-1}\left(e^{(\cot\theta)t}\right). \tag{4.8}$$

By the equality (15) it is seen that

$$\sinh u(t) = \frac{2\,e^{(\cot\theta)t}}{e^{2(\cot\theta)t} - 1}, \quad \cosh u(t) = \frac{e^{2(\cot\theta)t} + 1}{e^{2(\cot\theta)t} - 1}. \tag{4.9}$$

By writing the equalities (16) in (8) we obtain the parametrization (11) and this completes the proof (See, Figure 3). ■

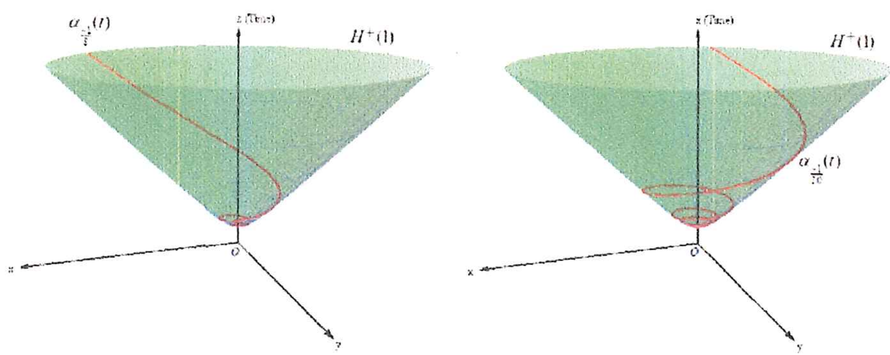

Figure 3: Two spirals on the surface of the hyperboloid model $H^{+}(1)$.

5 The Spirals on the Lorentzian Sphere $S_1^2(r)$

Since the Lorentzian sphere is a timelike surface, there are three types of geodesic called spacelike, timelike and lightlike (null) geodesics. Similarly, there must be three kinds of spirals that cut the meridians under fixed hyperbolic, timelike and Lorentzian angles. In this section, we find the differential equations of causal spirals that provide certain conditions and we give the parametrizations of the timelike, spacelike and lightlike spirals

which are the solutions of these equations.
We know that a parametrization of the Lorentzian sphere $S_1^2(r)$ is given by

$$y(u,v) = (r\,\mathrm{cosh}u\,\mathrm{cos}v, r\,\mathrm{cosh}u\,\mathrm{sin}v, r\,\mathrm{sinh}u) \tag{5.1}$$

where $0 \le v \le 2\pi$ and $u \in \mathbb{R}$. First, observe that

$$\begin{cases} \boldsymbol{y}_u(u,v) = (r\,\mathrm{sinh}u\,\mathrm{cos}v, r\,\mathrm{sinh}u\,\mathrm{sin}v, r\,\mathrm{cosh}u) \\ \boldsymbol{y}_v(u,v) = (-r\,\mathrm{cosh}u\,\mathrm{sin}v, r\,\mathrm{cosh}u\,\mathrm{cos}v, 0). \end{cases} \tag{5.2}$$

Hence,

$$E(u,v) = \langle \boldsymbol{y}_u, \boldsymbol{y}_u \rangle = -r^2,\ F(u,v) = \langle \boldsymbol{y}_u, \boldsymbol{y}_v \rangle = 0,\ G(u,v) = \langle \boldsymbol{y}_v, \boldsymbol{y}_v \rangle = r^2\mathrm{cosh}^2 u. \tag{5.3}$$

If $\boldsymbol{w}$ is a tangent vector to the sphere at the point $y(u,v)$, then $\boldsymbol{w}$ is given in the basis associated to $y(u,v)$ by

$$\boldsymbol{w} = a\boldsymbol{y}_u + b\boldsymbol{y}_v.$$

Thus, the square of the length of $\boldsymbol{w}$ is given by

$$|\boldsymbol{w}|^2 = I(\boldsymbol{w}) = Ea^2 + 2Fab + Gb^2 = -a^2r^2 + b^2r^2\mathrm{cosh}^2 u.$$

Let us determine the causal spirals in this coordinate neighborhood of the Lorentzian sphere $S_1^2(r)$ which make a constant angle θ with the meridians ($v = const.$). In this case, we give definitions as follows:

Definition 5.1 *Let $\beta_\lambda(t)$ be a differentiable curve on the surface of the Lorentzian sphere $S_1^2(r)$. The curve $\beta_\lambda(t)$ is called Lorentzian timelike (resp., spacelike, lightlike) spherical spiral if $\beta_\lambda(t)$ cuts all meridians of $S_1^2(r)$ with a constant hyperbolic (resp., timelike, Lorentzian) angle.*

Theorem 5.2 *The parametrization of all timelike (resp., spacelike, lightlike) spirals on the surface of the Lorentzian sphere $S_1^2(r)$ is given by*

$$\beta_\lambda(t) = (r\sec(\lambda t)\cos t, r\sec(\lambda t)\sin t, -r\tan(\lambda t)), \tag{5.4}$$

where $\lambda = \coth\theta$ (resp., $= \tanh\theta$, $= \pm 1$) and θ is constant hyperbolic (resp., timelike, Lorentzian) angle between meridians and the curve $\beta_\lambda(t)$.

Proof. We may assume that the required causal curve

$$\beta_\lambda(t) = y\,(u(t), v(t)) = (r\cosh u(t)\cos v(t), r\cosh u(t)\sin v(t), r\sinh u(t))$$

is the image by y of a curve $(u(t), v(t))$ of the uv˘plane. In this case, we have

$$\beta'_\lambda(t) = (r\,u'(t)\text{sinh}u(t)\cos v(t) - rv'(t)\cosh u(t)\sin v(t),\\ r\,u'(t)\text{sinh}u(t)\sin v(t) + rv'(t)\cosh u(t)\cos v(t), ru'(t)\cosh u(t))\,.$$

(i) Let the required causal curve $\beta_\lambda(t)$ be timelike curve.
At the point $y(u, v)$ where the timelike curve meets the meridians ($v = const.$), we have

$$-\cosh\theta = \frac{\langle \boldsymbol{y}_u, \beta'_\lambda(t)\rangle}{\|\boldsymbol{y}_u\|\,\|\beta'_\lambda(t)\|} = \frac{-u'}{\sqrt{(u')^2 - (v')^2\text{cosh}^2 u}}, \tag{5.5}$$

since in the basis $\{\boldsymbol{y}_u, \boldsymbol{y}_v\}$ the vector $\beta'_\lambda(t)$ has coordinates (u', v') and the vector $\boldsymbol{y}_u$ has coordinates $(1, 0)$. By (21) we obtain the differantial equation as

$$(u')^2\text{tanh}^2\theta - (v')^2\cosh^2 u = 0$$

or

$$\frac{du}{\text{cosh}u} = \pm\frac{dv}{\tanh\theta}. \tag{5.6}$$

Hence, by integration we obtain the differential equation of the Lorentzian spherical timelike spirals as

$$2\tan^{-1}\left(\tanh\frac{u}{2}\right) = \pm(v + c)\coth\theta, \tag{5.7}$$

where the constant of integration c is to be determined by giving one point $y(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the differential equation (23) gives

$$u(t) = 2\tanh^{-1}\left(\tan\left(\frac{\coth\theta}{2}t\right)\right). \tag{5.8}$$

By the equality (24) it is seen that

$$\sinh u(t) = -\tan(\lambda t), \ \cosh u(t) = \sec(\lambda t). \tag{5.9}$$

By writing the equalities (25) in (17) we obtain the parametrization (20) and this completes the proof of (i) (See, Figure 4).

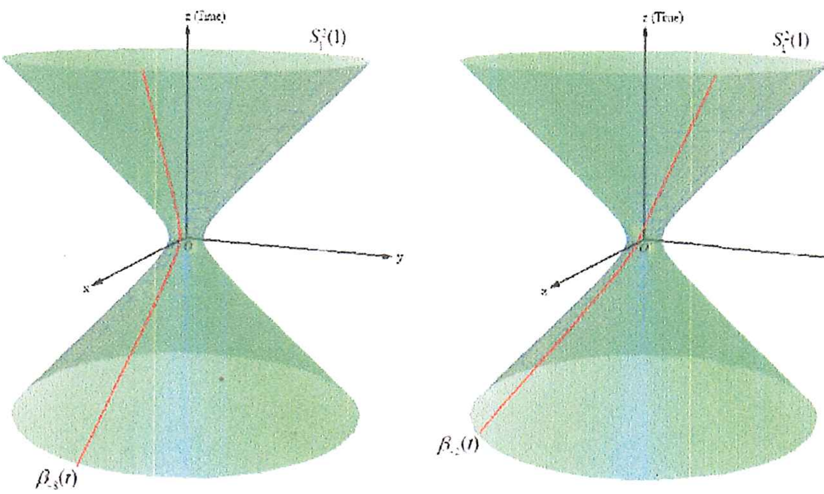

Figure 4: Two Timelike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

(ii) Let the required causal curve $\beta_\lambda(t)$ be spacelike curve.
At the point $y(u, v)$ where the timelike curve meets the meridians ($v = const.$), we have

$$\sinh\theta = \frac{|\langle \boldsymbol{y}_u, \boldsymbol{\beta}'_\lambda(t)\rangle|}{\|\boldsymbol{y}_u\| \ \|\boldsymbol{\beta}'_\lambda(t)\|} = \frac{u'}{\sqrt{-(u')^2 + (v')^2\cosh^2 u}}, \tag{5.10}$$

since in the basis $\{\boldsymbol{y}_u, \boldsymbol{y}_v\}$ the vector $\boldsymbol{\beta}'_\lambda(t)$ has coordinates (u', v') and the vector $\boldsymbol{y}_u$ has coordinates (1, 0). By (26) we obtain the differantial equation as

$$(u')^2\coth^2\theta - (v')^2\cosh^2 u = 0$$

or

$$\frac{du}{\cosh u} = \pm \frac{dv}{\coth\theta}. \tag{5.11}$$

Hence, by integration we obtain the differential equation of the Lorentzian spherical spacelike spirals as

$$2\tan^{-1}\left(\tanh\frac{u}{2}\right) = \pm(v+c)\tanh\theta, \tag{5.12}$$

where the constant of integration c is to be determined by giving one point $y(u_0, v_0)$ which the curve passes. By taking the signature $+$, $c = 0$ and $v(t) = t$, the solution of the equation (28) gives

$$u(t) = 2\tanh^{-1}\left(\tan\left(\frac{\tanh\theta}{2}t\right)\right). \tag{5.13}$$

By the equality (29) it is seen that

$$\sinh u(t) = -\tan(\lambda t), \ \cosh u(t) = \sec(\lambda t). \tag{5.14}$$

By writing the equalities (30) in (17) we obtain the parametrization (20) and this completes the proof of (ii) (See, Figure 5).

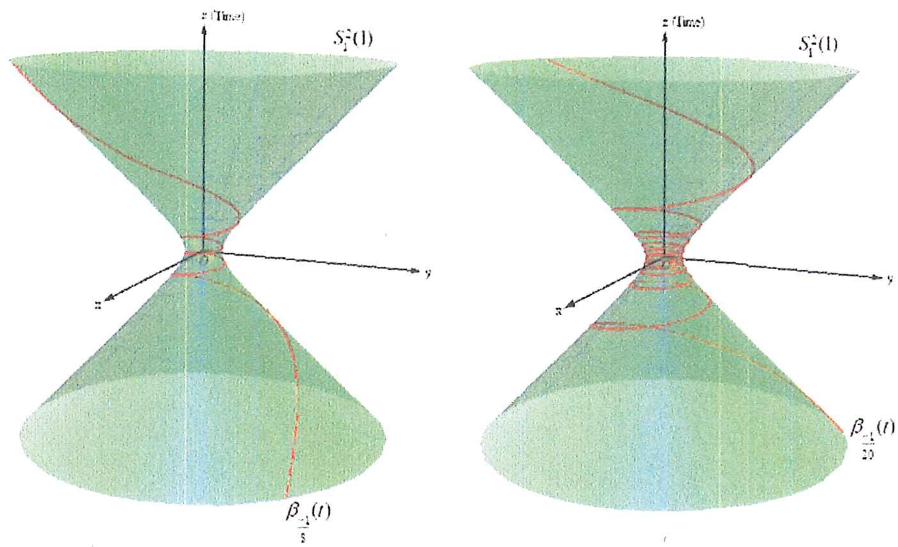

Figure 5: Two Spacelike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

(iii) Let the required causal curve $\beta_\lambda(t)$ be lightlike (null) curve. In this case, we have $\lambda = \pm 1$. Thus the parametrizations of lightlike spirals are given by

$$\beta_1(t) = (r, r\tan t, -r\tan t) \quad and \quad \beta_{-1}(t) = (r, r\tan t, r\tan t). \quad (5.15)$$

The parametrizations in (31) show two lightlike lines passing from the point $P(1,\ 0,\ 0)$. This means that all lightlike spirals on the Lorentzian sphere $S_1^2(r)$ are lightlike lines (geodesics) (See, Figure 6).

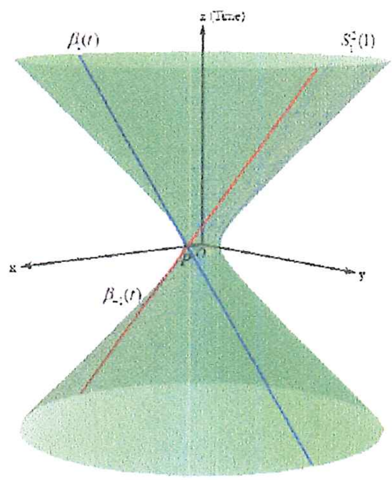

Figure 6: Two Lightlike spirals on the surface of the Lorentzian sphere $S_1^2(1)$.

Consequently, (i), (ii) and (iii) complete the proof. ■

6 The Spirals on the Lightlike Cone Λ^2

We know that a parametrization of the lightlike cone Λ^2 is given by

$$z(u,v) = (v\cos u, v\sin u, v) \tag{6.1}$$

where $0 \le u \le 2\pi$ and $v \in \mathbb{R}$. First, observe that

$$\begin{cases} z_u(u,v) = (-v\sin u, v\cos u, 0) \\ z_v(u,v) = (\cos u, \sin u, 1). \end{cases} \tag{6.2}$$

Hence,

$$E(u,v) = \langle z_u, z_u\rangle = v^2,\ F(u,v) = \langle z_u, z_v\rangle = 0,\ G(u,v) = \langle z_v, z_v\rangle = 0. \tag{6.3}$$

If w is a tangent vector to the lightlike cone at the point $z(u,v)$, then w is given in the basis associated to $z(u,v)$ by

$$w = az_u + bz_v.$$

Thus, the square of the length of $\boldsymbol{w}$ is given by

$$|\boldsymbol{w}|^2 = I(\boldsymbol{w}) = Ea^2 + 2Fab + Gb^2 = v^2a^2.$$

Let us determine the curves in this coordinate neighborhood of the lightlike cone Λ^2 which make a constant angle with the meridians (or parallels). In this case, we give definitions as follows:

Definition 6.1 *Let $\gamma_\lambda(t)$ be a differentiable curve on the surface of the lightlike cone Λ^2. The curve $\gamma_\lambda(t)$ is called lightlike conical spiral if $\gamma_\lambda(t)$ cuts all parallels (or lightlike lines) of Λ^2 with zero angle.*

Now, we give parametrization of the lightlike conical spirals with constant speed as follows:

Theorem 6.2 *The parametrization of all constant speed spirals on the surface of the lightlike cone Λ^2 is given by*

$$\gamma_\lambda(t) = (t\cos(\lambda \ln t), t\sin(\lambda \ln t), t). \tag{6.4}$$

where $\lambda \in \mathbb{R}^+$ denotes the constant speed of $\gamma_\lambda(t)$.

Proof. We may assume that the required curve

$$\gamma_\lambda(t) = z\,(u(t), v(t)) = (v(t)\cos u(t), v(t)\sin u(t), v(t))$$

is the image by z of a curve $(u(t), v(t))$ of the uvˇplane. In this case we have

$$\begin{aligned}\gamma_\lambda'(t) = (&v'(t)\cos u(t) - u'(t)v(t)\sin u(t),\\ &v'(t)\sin u(t) + u'(t)v(t)\cos u(t), v'(t)).\end{aligned}$$

At the point $z(u, v)$ where the curve meets the parallels ($v = const.$), we have

$$\frac{|\langle z_u, \gamma_\lambda'(t)\rangle|}{\|z_u\|\,\|\gamma_\lambda'(t)\|} = 1, \tag{6.5}$$

since in the basis $\{z_u, z_v\}$ the vector $\gamma_\lambda'(t)$ has coordinates (u', v') and the vector z_u has coordinates (1, 0). By (36) we satisfy (7), but (36) does not give any differential equation. On the other hand, we see that $\|\gamma'_\lambda(t)\| = u'(t)v(t)$. In this case, there is a way to obtain a differential equation: the spiral $\gamma_\lambda(t)$ can be thought as a constant speed curve. Thus, the inner product $|\langle z_u, \gamma'_\lambda(t)\rangle|$ becomes constant (for details, see [7]). Then, the differential equation is given by

$$u'v = \lambda$$

or

$$du = \lambda \frac{dv}{v}. \tag{6.6}$$

Hence, by integration we obtain the differential equation of the lightlike conical spiral as

$$u = \lambda \ln(v) + c, \tag{6.7}$$

where the constant of integration c is to be determined by giving one point $z(u_0, v_0)$ which the curve passes. Specially, we assume that $c = 0$ and $v(t) = t$. Thus, we have $u(t) = \lambda \ln t$. By writing these equalities in (32) we obtain the parametrization (35) and this completes the proof (See, Figure 7). ■

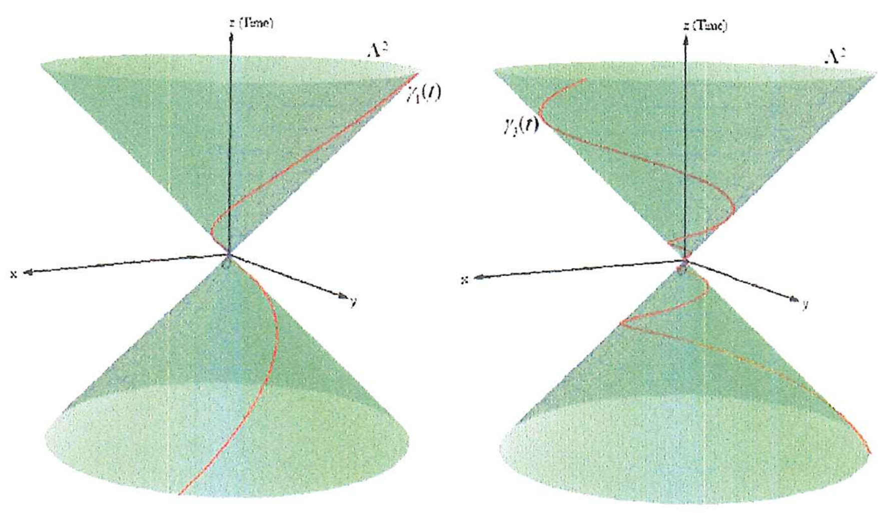

Figure 7: Two spirals on the surface of the lightlike cone Λ^2.

7 Conclusion

In this paper, we obtained differential equations of all spirals on the positive hyperbolic sphere $H^+(r)$, the Lorentzian sphere $S_1^2(r)$ and the positive lightlike cone Λ^+ of the Lorentz-Minkowski space $\mathbb{R}_1^3$, and gave the general parametrizations of these curves. In the next study, we will define spiral curves on some special surfaces of the space $\mathbb{R}_1^3$ and examine their geometries.

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