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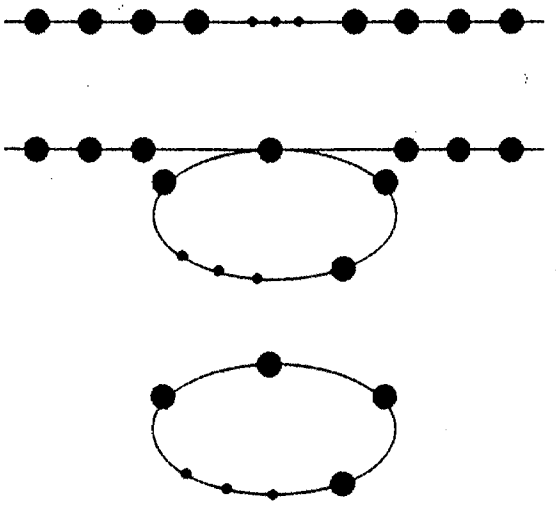
Hyperstructures and their Representations

T. Vougiouklis



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The drawing in the front page is related to the content of the monograph and presents graphically the H_V–group obtained by 'uniting' the integers 0 and n.

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CONTENTS

PREFACE iii

1. BASIC DEFINITIONS

1.1 Weak hyperstructures 1
1.2 The fundamental relations β^*, γ^*, ε^* 6
1.3 Special elements 18
1.4 H_b-structures 21
1.5 Some classes of H_v-structures 21

2. H_v-GROUPS

2.1 H_v-groups defined on the same set 29
2.2 H_v-groups and H_b-groups 37
2.3 Translations 41
2.4 Semidirect hyperproduct 43
2.5 Cyclicity 48

3. H_v-RINGS

3.1 Fundamental relations 51
3.2 H_v-rings, H_b-rings 57
3.3 Absorbing elements 61
3.4 H_v-modules 62
3.5 A matrix P-hyperalgebra 65

4. UNITING ELEMENTS

4.1 Uniting elements. Associativity, Reproductivity Commutativity 67
4.2 Other problems 72
4.3 Examples 75

5. H_v-HYPERGROUPOID ALGEBRA

5.1 Fundamental sets 77
5.2 H_v-hypergroupoid algebra 79

6. GENERALIZED PERMUTATIONS

6.1 The generalized permutations 85
6.2 The condition θ 91

7. REPRESENTATIONS BY GENERALIZED PERMUTATIONS

7.1 A non standard hyperoperation 97
7.2 H_v-group permutation representations 101

8. H_v-MATRIX REPRESENTATIONS

8.1 The problem 109
8.2 A necessary condition 114
8.3 Theorems 123
8.4 Representations by Boolean H_v-matrices 126

9. REDUCIBILITY AND CHARACTERS

9.1 Traces 131
9.2 Representations 135

APPENDIX 1 Classical hyperstructures

A1 Classical hyperstructures 141

APPENDIX 2 Very thin hyperstructures

A2.1 Very thin H_v-groups 147

A2.2 Very thin H_v-rings 151

APPENDIX 3 P-hyperstructures

A3.1 P-H_v-groups 157

A3.2 P-H_v-rings 163

A3.3 Cyclicity in P-hyperstructures 167

REFERENCES 173
SYMBOLS 177
INDEX 179

PREFACE

Travelling in the streets of the representation theory, one can see very beautiful landscapes! On the train of the generalized permutations and H_v-matrices one will inevitably pass through H_v-groups, H_v-rings, H_v-modules and other hyperstructures. One will enjoy the properties of those structures and will discover new ideas, terms and constructions. The mathematician-traveler will have to convey these algebraic hyperstrures by using maps and mainly homomorphisms.

The beauty of the journey is the reason for choosing this field; the lack of a systematic account of the results of this comparatively new concept prompted my decision to attempt the present research.

In the first three chapters the basic concepts of the field are presented. The uniting elements method is contained in the fourth chapter. In chapter five a H_v-hypergroupoid algebra is defined. The representations of H_v-groups by generalized permutations are the object of the sixth and seventh chapters. The main subject of this book is set forth in the last two chapters. There are also three appendices. The first which contains some classical hyperstructures is a short section since an excellent presentation of these structures can be found in Corsini's book "Prolegomena". The other two appendices contain two large classes of H_v-structures: the very thin and the P-H_v-structures.

There are samples of the author's poetry published in Greek and translated into English by J. Petropoulos.

Thanks are due to A. Dramalides for correcting the final version of this book.

I would like to thank my wife Dr Penelope Vougiouklis for her help on language and her patience; my deepest appologies to my daughters Eleni and Susanna for not being a good father this period.

Thomas Vougiouklis
Democritus University of Thrace

Ανασαίνοντας γεννήσαμε τη σκέψη
ανασαίνοντας τη μεταδόσαμε
εξακολουθούμε ν'ανασαίνουμε.

Breathing we gave birth to thought
breathing we handed it on
we continue to breathe.

CHAPTER 1

BASIC CONCEPTS

1.1 WEAK HYPERSTRUCTURES, H_v-STRUCTURES

In 1934, F.Marty introduced the hypergroup as a set H equipped with a hyperoperation $\cdot: H\times H \longrightarrow \mathcal{P}(H)$ which satisfies the associative law: $(xy)z=x(yz)$ for all x,y,z in H, i.e. the sets on both sides are equal, and the reproduction axiom is valid: $xH=Hx=H$ for all x in H. The motivating example was the following: Let G be a group and H any subgroup of G. Then $G/H = \{ xH: x\in G \}$ becomes a hypergroup where the hyperoperation is defined in a usual manner $\bar{x}\odot\bar{y} = \{ \bar{z}: z \in \bar{x}\cdot\bar{y} \}$ where $\bar{x}=xH$. Every hypergroup isomorphic to the above is called D-hypergroup and the class of D-hypergroups is not equal to the class of hypergroups. An extensive expository of the hypergroup theory, with a complete bibliography, can be found in "Prolegomena of hypergroup theory" (Corsini, 1993).

In this chapter a larger class than the known hyperstructures is defined. This new class is the object of this book and was introduced by Vougiouklis in the Fourth AHA congress, (1990).

1.1.1 DEFINITIONS

Let H be a set and $\cdot : H \times H \longrightarrow \mathcal{P}(H)$ be a hyperoperation. The $(\cdot)$ in H is called **weak associative**, we write **WASS**, if

$$(xy)z \cap x(yz) \neq \emptyset \quad \text{for all} \quad x,y,z \quad \text{in} \quad H.$$

The $(\cdot)$ is called **weak commutative**, we write **COW**, if

$$xy \cap yx \neq \emptyset \quad \text{for all} \quad x,y \quad \text{in} \quad H$$

The hyperstructure $(H,\cdot)$ is called **H_v-semigroup** if $(\cdot)$ is WASS. H is called **H_v-quasigroup** if the reproduction axiom is valid, i.e.

$$xH = Hx = H \quad \text{for all} \quad x \quad \text{in} \quad H.$$

H is called **H_v-group** if it is a H_v-quasigroup and a H_v-semigroup. It is called **H_v-commutative group** if it is a H_v-group and a COW. If the commutativity is valid, then H is called **commutative H_v-group.**

In an obvious way the **sub-H_v-group** of a H_v-group is defined.

Analogous definitions for other **H_v-structures**, as H_v-rings, H_v-fields, H_v-vector spaces and so on, can be given.

A hyperstructure $(R,+,\cdot)$ is called **H_v-ring** if

(a) $(+)$ and $(\cdot)$ are WASS,

(b) the reproduction axiom is valid for $(+)$, and

(c) $(\cdot)$ is **weak distributive** with respect to $(+)$, i.e.

$$\left.\begin{array}{l} x(y+z) \cap (xy + xz) \neq \emptyset \\ (x+y)z \cap (xz + yz) \neq \emptyset \end{array}\right\} \quad \text{for all} \quad x,y,z \text{ in } R$$

A H_v-ring can be COW with respect either to $(+)$ or to $(\cdot)$. If it is in both COW, we call it **COW H_v-ring**.

If the above (a) and (c) axioms are valid then R is called **H_v-semiring**. Similarly the H_v-near-ring is defined.

The definition of the H_v-field is given in the next paragraph.

PROPERTY
All the weak properties for hyperstructures can be applied for subsets. For example, if $(R,+,\cdot)$ is a COW H_v-ring then for all $A,B,C \subseteq R$ we have

$$(A+B)\cap(B+A) \neq \emptyset\ , \quad (AB)\cap(BA) \neq \emptyset\ , \quad A(B+C)\cap(AB+AC) \neq \emptyset\ .$$

To prove this, one has simply to take one element of each set.

A motivation to obtain H_v-structures is the following:

1.1.2 THE MOTIVATING EXAMPLE

Let $(G,\cdot)$ be a group, σ an equivalence relation in G and $\sigma(x)$ the σ-equivalence class of the element x in G. In G/σ consider the hyperoperation $\circ$ defined in the usual manner

$$\sigma(x)\circ\sigma(y) = \{\ \sigma(z)\ :\ z\in\sigma(x)\cdot\sigma(y)\ \} \quad \text{for all } x,y \text{ in } G.$$

Then $(G/\sigma,\circ)$ is a H_v- group which is not always a hypergroup.
Indeed, for all x,y,z in G we have

$$(xy)z \in (\sigma(x)\circ\sigma(y))\circ\sigma(z)$$

$$x(yz) \in \sigma(x)\circ(\sigma(y)\circ\sigma(z))$$

Thus $\circ$ is WASS. Moreover, the reproduction axiom is valid.

If for some x,y in G we have

$$\sigma(x)\circ\sigma(y) = \sigma(z) \quad \text{for all } z \in \sigma(x)\cdot\sigma(y)$$

then σ is said to satisfy the **fundamental property** for x and y. Consequently, if the fundamental property for all x,y in G is valid, then $\circ$ is an operation.

1.1.3 EXAMPLES

(i) Geometrical hyperoperations $(+)$ defined on a given incidence space A, have the property

$$\{a,b\} \subset a+b \quad \text{for all } a,b\in A.$$

Therefore $(A,+)$ is always a H_v-group, in fact it is a H_v-commutative group.

(II) Let V be a vector space over a field K. Define three hyperoperations in V as follows

$$\left.\begin{aligned} x\circledast y &= \{\ \lambda(x+y):\ 0\le\lambda\le 1\ \} \\ x\circledcirc y &= \{\ x+\lambda(y-x):\ 0\le\lambda\le 1\ \} \\ x\circledast y &= \{\ x+\lambda y:\ 0\le\lambda\le 1\ \} \end{aligned}\right\} \quad \text{for all } x,y\in V \text{ and } \lambda\in K.$$

Then one can obtain that for all $*,\blacksquare \in \{ \circledast,\circledcirc,\circledast \}$ the 9 hyperstructures $\langle V,*,\blacksquare\rangle$ are COW H_v-rings.

1.1.4 DEFINITIONS

Let $(H_1,\cdot)$, $(H_2,*)$ be two H_v-groups. A map $f: H_1 \longrightarrow H_2$ is called **H_v-homomorphism** or **weak homomorphism** or simply **homomorphism**, if

$$f(x\cdot y) \cap f(x)*f(y) \neq \emptyset \quad \text{for all } x,y \text{ in } H_1 .$$

f is called an **inclusion homomorphism** if

$$f(x\cdot y) \subset f(x)*f(y) \quad \text{for all } x,y \in H_1 .$$

Finally, f is called a **strong homomorphism** if

$$f(x\cdot y) = f(x)*f(y) \quad \text{for all } x,y \in H_1 .$$

If f is onto, 1:1 and strong homomorphism, then it is called **isomorphism**, if moreover f is defined on the same H_v-group then it is called **automorphism**. It is an easy verification that the set of automorphisms in H, written **Aut**H, is a group.

On a set G several H_v-structures can be defined. A partial order on those hyperstructures is introduced as follows:

1.1.5 DEFINITION

Let $(G,\cdot)$, $(G,*)$ be two H_v-groups, defined on the same set G. We call $(\cdot)$ **less** or **equal** than $(*)$, and write $\cdot\le *$, iff there is a $f\in Aut(G,*)$ such that

$$xy \subseteq f(x*y) \quad \text{for all } x,y \text{ in } G.$$

We also say that $(*)$ is **greater** than $(\cdot)$. Moreover we say that $(G,*)$ contains the $(G,\cdot)$. **Minimal** is called a H_v-group which contains no other H_v-group defined on the same set.

Following this definition, one can define the **union** $(*)$ of two hyperoperations $(+)$ and $(\circledast)$, and we write $* = +\cup\circledast$, defined on the same set H, as follows: $x*y = (x+y)\cup(x\circledast y)$ for all x,y in H.

Obviously, if $(+)$ or $(\circledast)$ is WASS (or COW) then the union hyperoperation is also WASS (or COW).

1.1.6 THEOREM

Greater hyperoperation from the one of a given H_v-group, defines a H_v-group. The weak commutativity is also valid for every greater hyperoperation. Analogous result can be proved for the other H_v-structures with the analogous weak propetries.

Proof
Obvious from the definition. ∎

We remark that this theorem is not true for hypergroups in the sense of Marty.

Notation. Let $(G,\cdot)$ be a hypergroupoid. An n-ary hyperproduct can be defined, induced by $(\cdot)$, by inserting n-2 parentheses in the sequence of elements $a_1,a_2,\ldots,a_n$ in a standard position. Let us denote by $p(a_1,\ldots,a_n)$ such a pattern of n-2 parentheses and by, P_n the set of all such patterns. For example, for n=4, we have,

$$P_4(a_1,a_2,a_3,a_4) =$$

$$= \left\{((a_1a_2)a_3)a_4,(a_1(a_2a_3))a_4,(a_1a_2)(a_3a_4),a_1((a_2a_3)a_4),a_1(a_2(a_3a_4))\right\}$$

Now let $(R,+,\cdot)$ be a H_v-ring then the n-2 parentheses in the sequence $a_1,\ldots,a_n$, n-1 hyperoperations are used which may be either $(+)$ or $(\cdot)$. Therefore, the set P_n is larger in H_v-rings than in hypergroupoids, since in each $p_i\in P_n$, for hypergroupoids, corresponds a set of elements p_{ij} where $j = 1,2,\ldots,2^{n-1}$ for H_v-rings. However, we still denote by the same symbol P_n the set of all such patterns.

1.1.7 DEFINITION

In any triple of elements (x,y,z) a parenthesis can be put in two ways, either (xy)z or x(yz). Call the first, **parenthesis on the left** and the second, **parenthesis on the right**. In a product of n elements $a_1,a_2,\ldots,a_n$ if all parentheses are on the left $(\ldots((a_1a_2)a_3)\ldots)a_n$ it is called **left form**, and when we have all parentheses on the right we have the **right form** $a_1(\ldots(a_{n-1}a_n)\ldots)$. In the terms of commutators the left form is called left-normed and the right form, right-normed. For a given pattern $p\in P_n$ one can obtain series of patterns $p_1,\ldots,p_\lambda$ so that p_λ has the right form

and for each $i\in\{1,\dots,\lambda-1\}$ the p_i has only one parenthesis on the left comparing with p_{i+1}. Moreover, if the a_j's are elements of a H_v-semigroup or H_v-ring R then we have

$$p_i(a_1,\dots,a_n)\cap p_{i+1}(a_1,\dots,a_n)\neq\emptyset\ ,\quad i\in\{1,\dots,\lambda-1\}\ ,\quad a_j\in R.$$

In the following in a H_v-ring the P_n will denote the set of patterns consisting of n-2 parentheses with respect to hypersum.

1.1.8 DEFINITIONS

Let $(R,+,\cdot)$ be a H_v-ring, and $n>1$ be an integer number. We call **n-ary circle hyperoperation**, and write $\circledast$, the one defined as follows

$$x_1\circledast x_2\circledast\dots\circledast x_n=\bigcup_{p\in P_n}p(x_1,x_2,\dots,x_n)\quad\text{for all } x_1,\dots,x_n\in R.$$

Remark that in general $(x_1\circledast\dots\circledast x_n)\circledast x_{n+1}\neq x_1\circledast\dots\circledast x_n\circledast x_{n+1}$

however, $(x_1\circledast\dots\circledast x_n)\circledast x_{n+1}\subset(x_1\circledast\dots\circledast x_n\circledast x_{n+1})$.

The above definitions can also be given for H_v-groups.

1.2 THE FUNDAMENTAL RELATIONS β^*, γ^*, ε^*

The **fundamental relation** β^*, is defined on $(H,\cdot)$ and the quotient H/β^* is semigroup, quasigroup, group or commutative group if H is H_v-semigroup, H_v-quasigroup, H_v-group or H_v-commutative group, respectively, called **fundamental** ones. A H_v-structure H is called **proper** if $H\neq H/\beta^*$, in other words there exists at least one product which is not singleton.

1.2.1 DEFINITION

Let $(H,\cdot)$ be a H_v-group. We define the β^* relation as the smallest equivalence relation, one can say also congruence, such that the quotient H/β^* is a group. The β^* is called **fundamental equivalence relation.**

Let us denote by $\mathcal{U}$ the set of all finite products of elements of H and define the relation β in H as follows:

$x\beta y$ iff $\{x,y\}\subset u$ where $u\in\mathcal{U}$.

1.2.2 THEOREM

The fundamental relation β^* is the transitive closure of the relation β.

Proof

Let's denote $\hat{\beta}$ the transitive closure of β and $\hat{\beta}(a)$ the class of the element $a \in H$. First, we prove that the quotient set $H/\hat{\beta}$ is a group. The product $\odot$ in $H/\hat{\beta}$ is defined in the usual manner

$$\hat{\beta}(a) \odot \hat{\beta}(b) = \{\hat{\beta}(c) : c \in \hat{\beta}(a) \cdot \hat{\beta}(b)\} \quad \text{for all } a,b \text{ in } H.$$

Take $a' \in \hat{\beta}(a)$, $b' \in \hat{\beta}(b)$. Then we have

$$a' \hat{\beta} a \quad \text{iff} \quad \exists\, x_1,\ldots,x_{m+1} \text{ with } x_1 = a',\ x_{m+1} = a \text{ and } u_1,\ldots,u_m \in \mathcal{U} \text{ such that } \{x_i, x_{i+1}\} \subset u_i,\ i=1,\ldots,m$$

and

$$b' \hat{\beta} b \quad \text{iff} \quad \exists\, y_1,\ldots,y_{n+1} \text{ with } y_1 = b',\ y_{n+1} = b \text{ and } v_1,\ldots,v_n \in \mathcal{U} \text{ such that } \{y_j, y_{j+1}\} \subset v_j,\ j=1,\ldots,n.$$

Therefore, we obtain

$$\left.\begin{array}{ll} \{x_i,\ x_{i+1}\} \cdot y_1 \subset u_i \cdot v_1 & ,\ i=1,\ldots,m-1 \\ x_{m+1} \cdot \{y_j,\ y_{j+1}\} \subset u_m \cdot v_j & ,\ j=1,\ldots,n \end{array}\right\} \qquad (1)$$

Therefore

$$u_i \cdot v_i = t_i \in \mathcal{U},\ i=1,\ldots,m-1 \quad \text{and} \quad u_m \cdot v_j = t_{m+j-1} \in \mathcal{U},\ j=1,\ldots,n$$

or $t_\kappa \in \mathcal{U}$ for all $\kappa \in \{1,\ldots,m+n-1\}$.

Now, pick up any elements $z_1,\ldots,z_{m+n}$ such that

$$z_i \in x_i \cdot y_i,\ i=1,\ldots,m \quad \text{and} \quad z_{m+j} \in x_{m+1} \cdot y_{j+1},\ j=1,\ldots,n,$$

and using the relation (1) we have

$$\{z_\kappa, z_{\kappa+1}\} \subset t_\kappa, \quad \kappa = 1,\dots,m+n-1.$$

So, every element $z_1 \in x_1 \cdot y_1 = a'b'$ is $\hat{\beta}$ equivalent to every element $z_{m+n} \in x_{m+1} \cdot y_{n+1} = ab$.

Thus $\hat{\beta}(a) \circ \hat{\beta}(b)$ is singleton so we can write

$$\hat{\beta}(a) \circ \hat{\beta}(b) = \hat{\beta}(c) \quad \text{for all} \quad c \in \hat{\beta}(a) \cdot \hat{\beta}(b).$$

Moreover, since $(\cdot)$ is WASS, it is obtained that $\circ$ is associative and consequently, $H/\hat{\beta}$ is a group.

Now let σ be an equivalence relation in H such that H/σ is a group. Denote $\sigma(a)$ the class of a. Then $\sigma(a) \circ \sigma(b)$ is singleton for all a,b in H, i.e.

$$\sigma(a) \circ \sigma(b) = \sigma(c) \quad \text{for all} \quad c \in \sigma(a) \cdot \sigma(b)$$

But also, for every $a,b \in H$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$, we have

$$\sigma(a) \circ \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

So this relation is valid for all finite products. That means that

$$\sigma(x) = \sigma(u) \quad \text{for every } u \in \mathcal{U} \text{ and for all } x \in u.$$

Thus for every $a \in H$,

$$x \in \beta(a) \text{ implies } x \in \sigma(a).$$

But σ is transitively closed, so we obtain

$$x \in \hat{\beta}(a) \text{ implies } x \in \sigma(a).$$

Thus, the relation $\hat{\beta}$ is the smallest equivalence relation in H such that $H/\hat{\beta}$ is a group, i.e. $\hat{\beta} = \beta^*$. ∎

The β^* equivalence relation was introduced on hypergroups by Koskas [18] and studied mainly by Corsini [5].

Since G/β^* is a group one can say that $(G,\cdot)$ is **by virtue** a group. This is why the hyperstructure $(G,\cdot)$ is called H_v-group.

Open problem

Freni in [11] proved that for hypergroups $\beta^* = \beta$. We do not know yet if this equality is valid for H_v-groups.

1.2.3 EXAMPLE

On the set Z_{mn} consider the hyperoperation $(\oplus)$ defined by setting

$$\bar{0}\oplus m = \{\bar{0}, m\} \quad \text{and} \quad x\oplus y = x+y \quad \text{for all} \quad (x,y)\in Z_{mn}^2 - \{(\bar{0}, m)\}.$$

Then $(Z_{mn}, \oplus)$ becomes a H_v-group. This is not associative since, for example, taking $k \notin mZ$ then

$$(\bar{0}\oplus m)\oplus k = \{\bar{0}, m\}\oplus k = \{k, m+k\} \quad \text{and} \quad \bar{0}\oplus(m\oplus k) = \bar{0}\oplus(m+k) = \{m+k\}$$

therefore, $\oplus$ is WASS but not associative. Moreover, it is COW but not commutative.

From the sum

$$\underbrace{[\ldots(\bar{0}\oplus m)\oplus\ldots\oplus m]}_{n-1 \text{ times}}\oplus m = \{\bar{0}, m, 2m, \ldots, (n-1)m\}$$

it is obtained that $\beta^*(\bar{0}) = \{\bar{0}, m, \ldots, (n-1)m\}$.

Similarly, for every $0<k<n-1$ we have

$$\underbrace{[\ldots(k\oplus m)\oplus\ldots\oplus m]}_{n-1 \text{ times}}\oplus m = \{k, k+m, \ldots, k+(n-1)m\}$$

So $\beta^*(k) = k+mZ$. That means that $\beta^* = \beta$ and

$$Z_{mn}/\beta^* \cong Z_m$$

1.2.4 EXAMPLE

Consider the group $(Z^n, +)$ and take $m_1, \ldots, m_n \in \mathbb{N}$. We define a hyperoperation $\oplus$ in Z^n as follows:

$$(m_1, \ldots, m_n)\oplus(0, \ldots, 0) = \{(m_1, \ldots, m_n), (0, \ldots, 0)\}$$

and $+=\oplus$ in the rest cases.

Then $(Z^n, \oplus)$ is a H_v-group which is WASS and COW. We have $\beta^*=\beta$ and for the fundamental group one can prove that

$$Z^n/\beta^* \cong Z_{m_1\cdots m_n}\times Z^{n-1}$$

This is obtained because for every $k\in\mathbb{N}$ we have

$$\underbrace{(m_1,\ldots,m_n)\oplus(\ldots\oplus((m_1,\ldots,m_n)\oplus(0,\ldots,0))\ldots)}_{k \text{ times}} =$$

$$= \{ (km_1,\ldots,km_n), ((k-1)m_1,\ldots,(k-1)m_n), \ldots,(0,\ldots,0) \}$$

Moreover, by adding, on the left, the element $(-km_1,\ldots,-km_n)$ the finite product

$$\{ (0,\ldots,0), \ldots,(-(k-1)m_1,\ldots,-(k-1)m_n), (-km_1,\ldots,-km_n) \}$$

it is obtained.

Therefore, for every $k\in Z$, we have

$(km,\ldots,km)\ \beta^*(0,\ldots,0)$.

1.2.5 EXAMPLE

Consider the group $(Z^n,+)$ and take $m_1,\ldots,m_n \in N$. We define a hyperoperation $\oplus$ in Z^n as follows:

$$\left.\begin{aligned}(m_1,0,\ldots,0)\oplus(0,0,\ldots,0) &= \{(m_1,0,\ldots,0),(0,0,\ldots,0)\}\\ (0,m_2,\ldots,0)\oplus(0,0,\ldots,0) &= \{(0,m_2,\ldots,0),(0,0,\ldots,0)\}\\ (0,0,\ldots,m_n)\oplus(0,0,\ldots,0) &= \{(0,0,\ldots,m_n),(0,0,\ldots,0)\}\end{aligned}\right\}$$

and $+=\oplus$ in the rest cases.

Then $(Z^n,\oplus)$ is a H_v-group which is WASS and COW. We have $\beta^*=\beta$ and for the fundamental group one can prove that

$$Z^n/\beta^* \cong Z_{m_1}\times Z_{m_2}\times\ldots\times Z_{m_n}$$

Exercise

Give explicit proofs for the isomorphisms of the above two examples.

1.2.6 DEFINITION

Let $(R,+,\cdot)$ be a H_v-ring. We define the γ^* relation as the smallest equivalence relation such that the quotient R/γ^* is a ring. The γ^* is called, also, **fundamental equivalence relation**, and R/γ^* is called **fundamental ring**.

Let us denote the set of all finite polynomials of elements of R, over $\mathbb{N}$, by $\mathcal{U}$.

Now, we define the relation γ as follows:

$$x\gamma y \quad \text{iff} \quad \{x,y\} \subset u \quad \text{where} \quad u\in\mathcal{U}.$$

1.2.7 THEOREM

The γ^* relation is the transitive closure of the relation γ.

Proof

We follow the same proof as in H_v-groups because it is important, and we omit this for the other H_v-structures.

Let $\hat{\gamma}$ be the transitive closure of γ, and denote by $\hat{\gamma}(a)$ the class of the element a. First we prove that the quotient set $R/\hat{\gamma}$ is a ring.

In $R/\hat{\gamma}$ both the sum $\oplus$ and the product $\odot$ are defined in the usual manner for all $a,b\in R$

$$\hat{\gamma}(a) \oplus \hat{\gamma}(b) = \{\hat{\gamma}(c) : c \in \hat{\gamma}(a) + \hat{\gamma}(b)\}$$

$$\hat{\gamma}(a) \odot \hat{\gamma}(b) = \{\hat{\gamma}(d) : d \in \hat{\gamma}(a) \cdot \hat{\gamma}(b)\}$$

Take $a'\in \hat{\gamma}(a)$, $b'\in \hat{\gamma}(b)$. But we have

$a'\hat{\gamma}a$ iff $\exists\, x_1,\dots,x_{m+1}$ with $x_1 = a'$, $x_{m+1} = a$ and

$u_1,\dots,u_m \in \mathcal{U}$ such that

$\{x_i,x_{i+1}\} \subset u_i$, $i=1,\dots,m$

and

$b'\hat{\gamma}b$ iff $\exists\, y_1,\dots,y_{n+1}$ with $y_1 = b'$, $y_{n+1} = b$ and

$v_1,\dots,v_n \in \mathcal{U}$ such that

$\{y_j,y_{j+1}\} \subset v_j$, $j=1,\dots,n$.

From the above we obtain

$$\left.\begin{array}{ll} \{x_i,\ x_{i+1}\} + y_i \subset u_i + v_i & ,\ i=1,\dots,m-1 \\ x_{m+1} + \{y_j,\ y_{j+1}\} \subset u_m + v_j & ,\ j=1,\dots,n \end{array}\right\} \qquad (1)$$

The sums

$$u_i + v_i = t_i, \quad i=1,\dots,m-1 \quad \text{and} \quad u_m + v_j = t_{m+j-1}, \quad j=1,\dots,n$$

are also polynomials, therefore, $t_\kappa \in \mathcal{U}$ for all $\kappa \in \{1,\dots,m+n-1\}$. Now, pick up elements $z_1,\dots,z_{m+n}$ such that

$$z_i \in x_i + y_i, \quad i=1,\dots,m \quad \text{and} \quad z_{m+j} \in x_{m+1} + y_{j+1}, \quad j=1,\dots,n,$$

and using the relation (1) we have

$$\{z_\kappa, z_{\kappa+1}\} \subset t_\kappa, \quad \kappa = 1,\dots,m+n-1.$$

Therefore, every element $z_1 \in x_1+y_1 = a'+b'$ is $\hat{\mathcal{V}}$ equivalent to every element $z_{m+n} \in x_{m+1}+y_{n+1} = a+b$.

Thus $\hat{\mathcal{V}}(a)\oplus\hat{\mathcal{V}}(b)$ is singleton; so we can write

$$\hat{\mathcal{V}}(a) \oplus \hat{\mathcal{V}}(b) = \hat{\mathcal{V}}(c) \text{ for all } c \in \hat{\mathcal{V}}(a)+\hat{\mathcal{V}}(b).$$

In a similar way it is proved that

$$\hat{\mathcal{V}}(a) \odot \hat{\mathcal{V}}(b) = \hat{\mathcal{V}}(d) \quad \text{for all} \quad d \in \hat{\mathcal{V}}(a)\cdot\hat{\mathcal{V}}(b).$$

The weak associativity and the weak distributivity on R guarantee that the associativity and distributivity are valid for the quotient $R/\hat{\mathcal{V}}$. Therefore, $R/\hat{\mathcal{V}}$ is a ring.

Now let σ be an equivalence relation in R such that R/σ is a ring. Denote $\sigma(a)$ the class of a. Then $\sigma(a)\oplus\sigma(b)$ and $\sigma(a)\odot\sigma(b)$ are singletons for all a,b in R, i.e.

$$\sigma(a)\oplus\sigma(b) = \sigma(c) \quad \text{for all} \quad c \in \sigma(a)+\sigma(b)$$

$$\sigma(a)\odot\sigma(b) = \sigma(d) \quad \text{for all} \quad d \in \sigma(a)\cdot\sigma(b)$$

Thus we can write, for every $a,b\in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a)\oplus\sigma(b) = \sigma(a+\beta) = \sigma(A+B), \quad \sigma(a)\odot\sigma(b) = \sigma(a\beta) = \sigma(A\cdot B).$$

By induction, we extend these relations on finite sums and products. So for every $u\in\mathcal{U}$ we have, for all $x\in u$, the relation

$$\sigma(x) = \sigma(u)$$

Therefore, for every $a\in R$,

$$x\in\gamma(a) \text{ implies } x\in\sigma(a).$$

But σ is transitively closed, so we obtain

$$x\in\hat{\gamma}(a) \text{ implies } x\in\sigma(a).$$

That means that $\hat{\gamma}$ is the smallest equivalence relation in R such that $R/\hat{\gamma}$ is a ring, i.e. $\hat{\gamma} = \gamma^*$. ∎

We remark that the set $\mathcal{U}$ is not simplified as in hyperrings. There, this set consists of elements which are finite sums of finite products because of the inclusion distributivity.

1.2.8 DEFINITIONS

A H_v-ring is called **H_v-skew field** if its fundamental ring is a skew field, and it is called **H_v-field** if its fundamental ring is a field.

The fundamental relations can be defined on other classical H_v-structures. For example:

M is a **left H_v-module** over a H_v-ring R if $(M,+)$ is a COW H_v-group, and there exists

$$\cdot: R\times M \longrightarrow \mathcal{P}(M) : (a,x) \longmapsto ax$$

such that, for all a,b in R and x,y in M, we have:

$$a(x+y) \cap (ax+ay) \neq \varnothing , \quad (a+b)x \cap (ax+bx) \neq \varnothing , \quad (ab)x \cap a(bx) \neq \varnothing.$$

The **fundamental relation** ε^* in M over R, can be defined as the smallest equivalence relation such that the quotient M/ε^* to be a module over the ring R/γ^* . The same definition can be given for the **H_v-vector spaces**.

All the canonical maps φ from a H_v-structure to their fundamental structure are called **fundamental maps**.

If $\varphi: H \longrightarrow H/\beta^*$ is the fundamental map of a H_v-group then the kernel of φ is called **core** and it is denoted by ω_H . Therefore, $\omega_H = \{x\in H : \varphi(x)=1\}$, where 1 denotes the unit of the fundamental group. Moreover, H is called $|\omega_H|$-H_v-group. The 1-hypergroups, especially the feebly canonical ones, have been mainly investigated by Corsini.

Although the following theorem is valid more generally, we restrict ourselves on H_v-groups.

1.2.9 THEOREM

Let $(H,\cdot)$ be H_v-group, then

$$x\beta^*x' \quad \text{iff} \quad \text{there exist } A,A' \subset \beta^*(a),\ B,B' \subset \beta^*(b),$$

$$\text{for some } a,b \text{ in } H, \text{ such that}$$

$$xA \cap B \neq \emptyset \qquad x'A' \cap B' \neq \emptyset$$

Proof

Let $\varphi: H \longrightarrow H/\beta^*$ be the canonical map. Suppose that there are $a,b\in H$ and $A,A'\subset\beta^*(a)$, $B,B'\subset\beta^*(b)$ such that $xA\cap B\neq\emptyset$ and $x'A'\cap B'\neq\emptyset$, then we have

$$\varphi(x)\circ\varphi(A) \cap \{\varphi(B)\} \neq \emptyset \quad \text{and} \quad \varphi(x')\circ\varphi(A') \cap \{\varphi(B')\} \neq \emptyset$$

But $\circ$ is actually a product so we obtain

$$\beta^*(x)\circ\beta^*(a) = \beta^*(b) \quad \text{and} \quad \beta^*(x')\circ\beta^*(a) = \beta^*(b)$$

which implies $\beta^*(x) = \beta^*(b)\circ(\beta^*(a))^{-1} = \beta^*(x')$, thus $x\beta^*x'$.

The converse is immediate because if $x\beta^*x'$ then we can simply take $A = A' = \omega_H$ and $B = B' = \beta^*(x)$. ■

1.2.10 THEOREM

Let $(H,\cdot)$ be a H_v-group. Then $u\in\omega_H$ if and only if there exists $a\in H$ and $A\subset\beta^*(a)$ such that $uA \cap A \neq \emptyset$

Proof

Let $u\in\omega_H$. Take $A=\beta^*(a)$ for any $a\in H$, then

$$\varphi(u\beta^*(a)) = \varphi(u)\varphi(\beta^*(a)) = \beta^*(a)$$

thus $u\beta^*(a) \subset \beta^*(a)$ so $uA \cap A \neq \emptyset$.

Conversely, from $uA \cap A \neq \emptyset$ we obtain that $\varphi(uA) = \varphi(A)$ or $\varphi(u)=\omega_H$ so $u \in \omega_H$. ■

We can define the fundamental relations on equivalence relations as well. One can describe the situation as follows:

1.2.11 DEFINITION

Let $(S,\cdot)$ be a semigroup, σ be an equivalence relation in S and let us denote by $\sigma(x)$ the σ-equivalence class of $x\in S$. In S we define the **μ-equivalence relation on σ** as follows:

$a\theta b$ iff $\exists\ x_0,x_1,\ldots,x_r\ ;\ u_{i_1},\ldots,u_{i_r} \in S$ with $a=x_0$, $b=x_r$

and $I_1,\ldots,I_r$ finite sets of indices such that

$$\{x_{j-1},x_j\} \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j}) \quad , \quad j \in \{1,\ldots,r\}$$

Let us denote $\mu(x)$ the μ-class of x. Then we have:

1.2.12 THEOREM

Every μ-equivalence relation is a congruence i.e. the fundamental property on the equivalence classes is satisfied.

Proof

We want to prove that for all a,b in S the sets

$$\mu(a)\circ\mu(b) = \{\, d : d \in \mu(a)\cdot\mu(b) \,\}$$

are singletons.

We suppose that $a_1,a_2\in\mu(a)$, $b_1,b_2\in\mu(b)$ then we shall prove that the elements $d_1=a_1b_1$, $d_2=a_2b_2$ are μ-equivalent.

Since $a_1, a_2 \in \mu(a)$ or $a_1\mu a_2$ we have

$$\exists\ x_0,x_1,\ldots,x_r;\ u_{i_1},\ldots,u_{i_r} \in S \text{ with } a_1=x_0,\ a_2=x_r \text{ and}$$

$$I_1,\ldots,I_r \text{ finite sets of indices, such that}$$

$$\{x_{j-1},x_j\} \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j}) \qquad j\in\{1,\ldots,r\} \tag{1}$$

Moreover, since $b_1,b_2\in\mu(b)$ or $b_1\mu b_2$, we have

$$\exists\ y_0,y_1,\ldots,y_t;\ v_{p_1},\ldots,v_{p_t} \in S \text{ with } b_1=y_0,\ b_2=y_t \text{ and}$$

$$P_1,\ldots,P_t \text{ finite sets of indices, such that}$$

$$\{y_{q-1},y_q\} \subset \cdot \prod_{p_q \in P_q} \sigma(v_{p_q}) \qquad q\in\{1,\ldots,t\} \tag{2}$$

From (1) we obtain, for $i\in\{1,\ldots,r\}$,

$$\{x_{j-1}b_1, x_jb_1\} \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j})b_1 \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j})\sigma(b_1)$$

and from (2) we obtain, for $q \in \{1, \ldots, t\}$

$$\{a_2y_{q-1}, a_2y_q\} \subset a_2 \cdot \prod_{p_q \in P_q} \sigma(v_{p_q}) \subset \sigma(a_2) \cdot \prod_{p_q \in P_q} \sigma(v_{p_q})$$

But we have $a_2b_1 = x_rb_1 = a_2y_0 = a_2b_2$ and $x_0b_1 = a_1b_1$, $a_2y_t = a_2b_2$.

Therefore, from the definition of μ equivalence on σ, we have

$$(a_1b_1)\mu(a_2b_2) \quad \text{or} \quad d_1\mu d_2 \quad \text{for all} \quad d_1 \in a_1b_1 \,,\; d_2 \in a_2b_2$$

So the fundamental property for the hyperproduct $\circ$ was proved:

$$\mu(a) \circ \mu(b) = \mu(w) \quad \text{for all} \quad w \in \mu(a) \cdot \mu(b). \qquad \blacksquare$$

1.2.13 THEOREM

Let $(S, \cdot)$ be a semigroup (or groupoid), σ be an equivalence relation in S. Take the canonical maps

$$\varphi_1 : S \longrightarrow S/\sigma, \quad \varphi_2 : S/\sigma \longrightarrow (S/\sigma)/\beta^*, \quad \varphi : S \longrightarrow S/\mu.$$

Then

$$\langle S/\mu, \circ \rangle \cong \langle (S/\sigma)/\beta^*, \bullet \rangle$$

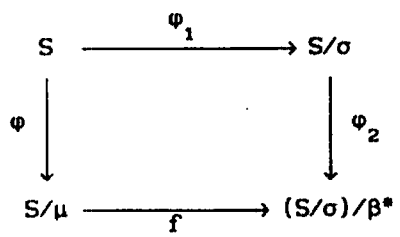
i.e. the μ relation on σ is the minimal congruence containing σ.

Proof

Consider the μ-equivalence relation on σ and the map

$$f : \mu(a) \longmapsto \beta^*(\sigma(a))$$

Then we have to prove that f is an isomorphism in the commutative diagram



First we prove that f is well defined. Take $b \in \mu(a)$ so

$a\theta b$ iff $\exists\ x_0, x_1, \ldots, x_r ; u_{i_1}, \ldots, u_{i_r} \in S$ with $a=x_0$, $b=x_r$

and $I_1, \ldots, I_r$ finite sets of indices such that

$$\{ x_{j-1} , x_j \} \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j}) \quad , \quad j \in \{ 1, \ldots, r \}$$

But, by definition, we have for $j\in\{1,\ldots,r\}$

$$\circledcirc \prod_{i_j \in I_j} \sigma(u_{i_j}) = \{ \sigma(w) : w \in \cdot \prod_{i_j \in I_j} \sigma(u_{i_j}) \}$$

Therefore,

$$\{ \sigma(x_{j-1}), \sigma(x_j) \} \subset \circledcirc \prod_{i_j \in I_j} \sigma(u_{i_j}) \quad , \quad j\in\{1,\ldots,r\}$$

from which we obtain $\sigma(a)\beta^*\sigma(b)$ or $\sigma(b)\in\beta^*(\sigma(a))$.

Thus f is well defined.

Moreover, f is a homomorphism because, from the fundamental property for μ and β^*, we have for $\mu(a),\mu(b) \in S/\mu$,

$f(\mu(a)\circledcirc\mu(b)) = \beta^*(\sigma(\mu(a)\circledcirc\mu(b))) = \beta^*(\sigma(\mu(w)))$ for all $w\in\mu(a)\cdot\mu(b)$

and $\beta^*(\sigma(a))\circledast\beta^*(\sigma(b)) = \beta^*(v)$ for all $v\in\sigma(a)\cdot\sigma(b)$.

Thus setting $w=v=ab$, it is obtained that f is a homomorphism.

It remains to prove that f is 1:1. To do this is enough to prove that $\sigma(a)\beta^*\sigma(b) \longrightarrow a\mu b$. Suppose $\sigma(a)\in\beta^*(\sigma(b))$, then

$\exists\ \sigma(x_0),\ldots,\sigma(x_r);\sigma(u_{i_1}),\ldots,\sigma(u_{i_r}) \in S/\sigma$ with $\sigma(a)=\sigma(x_0),\sigma(b)=\sigma(x_r)$

and $I_1,\ldots,I_r$ finite sets of indices such that

$$\{ \sigma(x_{j-1}) , \sigma(x_j) \} \subset \circledcirc \prod_{i_j \in I_j} \sigma(u_{i_j}) \quad , \quad j \in \{1\ldots,r \} .$$

Therefore, for each j, there exist $x'_{j-1}\in \sigma(x_{j-1})$, $x'_j \in \sigma(x_j)$ such that

$$\{ x'_{j-1},\ x'_j \} \subset \cdot \prod_{i_j \in I_j} \sigma(u_{i_j}) \quad , \quad j \in \{1,\ldots,r\}$$

Take $x'' \in \sigma(x_j)$ and consider the relations

$$\{ x'_j,\ x'_{j+1} \} \subset \cdot \prod_{i_{j+1} \in I_{j+1}} \sigma(u_{i_{j+1}}) \quad , \quad j \in \{1,\ldots,r\}$$

such that $\{ x'_j,\ x'' \} \subset \sigma(x_j)$.

Putting all the above relations together we have a set of finite indices $L_1,\ldots,L_k$ such that

$$\exists\ y_o, y_1, \ldots, y_k\ ;\ g_{\ell_1}, \ldots, g_{\ell_k} \in S \quad \text{with} \quad a=y_o,\ b=y_k \quad \text{such that}$$

$$\{ y_{j-1}\ ,\ y_j \} \subset \cdot \prod_{\ell_j \in L_j} \sigma(g_{\ell_j}) \quad , \quad j \in \{1,\ldots,k\}\ .$$

So $b \in \mu(a)$. Therefore, f is 1:1. ∎

1.3 SPECIAL ELEMENTS

1.3.1 DEFINITIONS

An element u of a H_v-structure H is caled **scalar** with respect to a hyperoperation $(*)$ if $u*x = x*u =$ singleton for all $x \in H$. An element $s \in H$ is called **single** if $\beta^*(s) = \{s\}$ (similarly $\gamma^*(s)=\{s\}$, $\varepsilon^*(s)=\{s\}$ and so on). We denote by S_H the set of all single elements. Remark that if $x \in S_H$ then $x \in \Pi a_i$ implies $\Pi a_i = \{x\}$ where $a_i \in H$. If a H_v-group has $S_H \neq \emptyset$ then one can find the fundamental classes using the following:

1.3.2 THEOREM

Let $(H,\cdot)$ be a H_v-group and $s \in S_H$. Let $a \in H$, take any element $v \in H$ such that $s \in av$, then

$$\beta^*(a) = \{ h \in H : hv = s \}$$

and the core of H is

$$\omega_H = \{ u \in H:\ us=s \} = \{ u \in H:\ su=s \}.$$

Proof

We have $s \in av$ so $s = av$ which means $s = \beta^*(a)\cdot\beta^*(v)$. Thus for all $h \in \beta^*(a)$ we have $hv = s$.

Conversely, let $s = hv$, then $s = \beta^*(h)\cdot\beta^*(v)$. Since H/β^* is a group, we have $\beta^*(h) = s(\beta^*(v))^{-1} = \beta^*(a)$ so $h \in \beta^*(a)$.

The proof for the core is obvious ∎

Remark that Theorem 1.3.2 can be applied for H_v-rings in a similar manner as well.

1.3.3 THEOREM

Let $(H,\cdot)$ be a H_v-group and $s \in S_H \neq \emptyset$, then

$$sx = \beta^*(sx) \quad \text{and} \quad xs = \beta^*(xs) \quad \text{for all } x\in H.$$

Proof

Let us suppose the absurd. For some x there exist $t\in sx$, and an element $t'\in \beta^*(t)$ such that $t'\notin sx$. From the reproductivity there exists $v\in H$, $v\neq s$ such that $vx \ni t'$. Then we have $\beta^*(v)\beta^*(x) = \beta^*(t')$, and on the other hand $\beta^*(s)\beta^*(x) = \beta^*(t)$, so $\beta^*(v)\beta^*(x) = \beta^*(s)\beta^*(x)$, thus $\beta^*(v) = \beta^*(s)$. But $s\in S_H$, so $\beta^*(s) = \{s\}$, thus $v=s$ which is absurd. ∎

This theorem proves that the product of a single element with any element is always a whole fundamental class. Moreover, if the H_v-group is proper then every $s\in S_H$ is not scalar, (cf the very thin hypergroups).

Similarly, one can prove that if $(R,+,\cdot)$ be a H_v-ring and $s\in S_R \neq \emptyset$, then $s+x = \gamma^*(s+x)$, $x+s = \gamma^*(x+s)$ for all $x\in R$, however, there is no analogous relation for the hyperproduct.

1.3.4 REMARKS

Suppose that $(H,\cdot)$ is a H_v-group with $S_H\neq\emptyset$ then the only greater hyperoperations $*>\cdot$ for which the H_v-groups $(H,*)$ contain single elements are the ones with the same fundamental group. This is so since the fundamental classes are determined from the products of a single element with the elements of the group. On the other side, a less hyperoperation $\circ<\cdot$ can have the same set S_H if only in the products of non-single elements the $\circ$ is less than $\cdot$.

Finally, if σ and σ' be equivalent relations with $\sigma<\sigma'$ such that H/σ and H/σ' be groups non equal, then they can not have both single elements.

1.3.5 COROLLARY

Let $(H,\cdot)$ be a H_v-group. If $S_H\neq\emptyset$ then $\beta^* = \beta$.

Proof

Obvious, since all the β^* classes can be obtained as products of two elements one of which is single. ∎

1.3.6 DEFINITION

An element x of a H_v-group $(H,\cdot)$ will be called **left complete** if $xy=\beta^*(xy)$ for all $y\in H$. Similarly will be called **right complete** if $yx=\beta^*(yx)$ for all $y\in H$, and if x is left and right complete then it will be called **complete element**. Thus, if a H_v-group has all its elements complete, then it is complete in the sense of Corsini [5]. Remark that, from Theorem 1.3.3, all single elements are complete.

1.3.7 DEFINITION

Let a H_v-structure H is partitioned into equivalence classes $\sigma(x)$. Then we shall say that the **reproductivity of classes** is valid, and we write **RC**, with respect to a hyperoperation $(\cdot)$ if

$$x\sigma(y) = \sigma(xy) = \sigma(x)y \quad \text{for all } x,y\in H.$$

This property is obviously valid for the partition of a H_v-group into the fundamental classes.

1.3.8 THEOREM

Let $(R,+,\cdot)$ be H_v-ring containing a zero element 0 (resp. a unit 1), then if $0\in S_R$ (resp. if $1\in S_R$) implies 0 (resp. 1) is unique.

Proof

Let $0'$ be also zero, then $0'\in 0+0'=0$ so $0'=0$.
Similar proof for the 1. ∎

1.3.9 DEFINITION

A H_v-ring containing the elements zero (0) and unit (1) which are singles, will be called **H_v-single ring**. Similarly, a H_v-group will be called **H_v-single group** if it contains a single unit element.

1.4 H_b-STRUCTURES

Now we define a very large subclass of H_v-structures as follows:

1.4.1 DEFINITION

A H_v-structure (as H_v-group, H_v-ring, H_v-vector space etc) is called **H_b-structure** if its hyperoperations contain operations which define a corresponding structure (as group, ring, vector space etc, respectively). In a H_b-structure one contained structure will be called **b-structure** and its operations are called **b-operations**. Obviously, any H_b-structure may contain more than one b-structures.

The Examples 1.2.3, 1.2.4 and 1.2.5 are examples of H_b-groups. P-H_v-groups with $1\in P$, are obviously H_b-groups, see Appendix 3.

1.4.2 THEOREM

The proper H_b-groups have no single elements.

Proof

Let $(H,\cdot)$ be H_b-group which contains the b-group $(H,*)$, let $|\beta^*(a)|>1$ and $a'\neq a$ such that $a'\in\beta^*(a)$. We suppose on the contrary that $s\in S_H\neq\emptyset$. Then there exists $b\in H$ such that $ab\ni s$ so $a*b=s$. Moreover we must have $a'*b=s$. But in $(H,*)$ there exists a unique inverse of b, say b^{-1}, for which we obtain $a=s*b^{-1}=a'$ which absurds our hypothesis, thus $S_H=\emptyset$. ∎

1.5 SOME CLASSES OF H_v-STRUCTURES

In this paragraph some constructions of H_v-structures, mainly H_v-groups and H_v-rings, are presented. These constructions are motivated from the properties of the single and scalar elements thus they are denoted by the letter S, and can be useful in the representation theory of H_v-groups by H_v-matrices.

1.5.1 CONSTRUCTION S_1

Let $(H,\cdot)$ be a H_v-group and H/β^* its fundamental group. A H_v-group with a single element $s\in S_H$, with the same fundamental group, can be constructed as follows:

Consider the set $H'= H-(\beta^*(s)-\{s\})$ and define a hyperoperation $(*)$ by putting

$$x*y = \begin{cases} \beta^*(xy) & \text{if} \quad s \in \{x,y\} \\ s & \text{if} \quad xy \subset \beta^*(s) \\ xy & \text{otherwise} \end{cases}$$

Then $(H',*)$ is H_v-group where $s\in S_H$ and $H'/\beta^* \cong H/\beta^*$.

Proof
Let $x,y,z \in H'$. If $s \in \{x,y,z\} \cup xy \cup yz$, then, from the reproductivity of the fundamental classes, it is obtained

$$(x*y)*z = x*(y*z) = \beta^*((xy)z) = \beta^*(x(yz)).$$

In the rest of the cases $(x*y)*z = (xy)z$, $x*(y*z) = x(yz)$ so

$$(x*y)*z \cap x*(y*z) \neq \emptyset, \quad \text{thus} \quad (*) \quad \text{is WASS.}$$

The rest of the proof is obvious. ∎

Remark that if we choose an element of the core $\omega_{H'}$, then H' becomes an 1-H_v-group.

Obviously one can apply S_1 consequently in order to obtain more single elements.

Another straightforward proof of the construction S_1, which also can be used for similar consructions, is given as follows:

Second proof of S_1
Consider the hyperoperation $(\blacksquare)$ defined in H by putting

$$x\blacksquare y = \begin{cases} \beta^*(xy) & \text{if} \quad s\in\{x,y\} \quad \text{or} \quad xy \subset \beta^*(s) \\ xy & \text{otherwise} \end{cases}$$

The $(\blacksquare)$ is greater than $(\cdot)$, so $(H,\blacksquare)$ is a H_v-group. Then consider

the projection $p : H \longrightarrow H'$ defined by

$$p(\beta^*(s)) = s \quad \text{and} \quad p(x) = x \quad \text{for all} \quad x \in H-\beta^*(s).$$

Then in H' $(\blacksquare)$ coincides with $(*)$, so $(H',*)$ is a H_v-group. ■

1.5.2 CONSTRUCTION S_2

Let $(R,+,\cdot)$ be a H_v-ring (or H_v-field) and R/γ^* its fundamental ring (resp. field). Take $s \in R$ and consider the set $R' = R-(\gamma^*(s)-\{s\})$. In R' we define the hyperoperations $(\oplus)$ and $(\odot)$ by setting

$$x \oplus y = \begin{cases} \gamma^*(x+y) & \text{if} \quad s \in \{x,y\} \\ s & \text{if} \quad x+y \subset \gamma^*(s) \\ x+y & \text{otherwise} \end{cases} \qquad x \odot y = \begin{cases} \gamma^*(xy) & \text{if} \quad s \in \{x,y\} \\ s & \text{if} \quad xy \subset \gamma^*(s) \\ xy & \text{otherwise} \end{cases}$$

Then $(R',\oplus,\odot)$ is H_v-ring (resp. H_v-field) with s single and

$$R'/\gamma^* \cong R/\gamma^* .$$

Proof

It is straightforward by enlarging the given hyperoperations, up to the fundamental class of s, and then taking the projection as in the second proof of the Contruction S_1 . ■

1.5.3 REMARK

Consider H_v-rings whose fundamental rings R/γ^* have unit element $\bar{1}$. From those H_v-rings one can take an element of the zero fundamental class $0 \in \bar{0}$, and one of the unit fundamental class $1 \in \bar{1}$. Then by applying the Contruction S_2 for both elements $0,1$ a H_v-ring is obtained which has 0 and 1 single elements which, moreover, are mutually scalars. We have the same remark for H_v-fields.

A construction of a H_v-group such that to have given fundamental group is the following:

1.5.4 CONSTRUCTION S_3

Let $(G,\cdot)$ be a group, and let H be a set which is partitioned into $|G|$ equivalence classes of which, a chosen element s is alone in its own class. Index the classes of H with the elements of G, so

$x \longrightarrow H_x$ for each $x \in G$ and $H = \bigcup_{x \in G} H_x$.

Since the class of the element s is singleton, we denote by the same letter the corresponding element of G, so $s := H_s = \{s\}$. From each class $H_x = \{h_x^i : i \in I\}$, we choose and fix an element h_x^a . Now we define a hyperproduct (*) by setting

$$h_x^i * h_y^j = \begin{cases} H_{xy} & \text{if } s \in \{x, y, xy\} \\ \\ \text{is RC and } h_{xy}^a \in h_x^i * h_y^j & \text{otherwise} \end{cases}$$

Then $(H, *)$ is H_v-group with $H/\beta^* \cong G$ where s is single.

Proof

From the definition the reproductivity of classes is valid so H is a H_v-group. Moreover, the condition $h_{xy}^a \in h_x^i * h_y^j$ guarantees that (*) is WASS. Thus $(H, *)$ is H_v-group and the rest of the proof is obvious. ■

1.5.5 CONSTRUCTION S_4

A standard procedure, more simple and explicit, which is a special case of the Construction S_3 , is given as follows:

$$h_x^i * h_y^j = \begin{cases} H_{xy} & \text{if } a \in \{i, j\} \\ \\ h_{xy}^a & \text{otherwise} \end{cases}$$

One simply has to remark that the chosen element of H_s is always the element s, and since

$$h_x^a * h_y^j = H_{xy} = h_x^i * h_y^a$$

the reproductivity of classes is valid.

In the S_4 if one take the sets H_x, instead of the h_x^a , then the greatest hyperoperation of these constructions is obtained. This is similar to the construction of (H,G)-hypergroups, see below 1.5.13.

1.5.6 REMARK

The Construction S_4 can be applied for two or more elements and for more hyperoperations but, in order that the weak axioms would be valid, one has to use always the same fixed elements. Thus we can apply this remark for H_v-rings or H_v-fields.

1.5.7 EXAMPLE

Let us consider the field $Z_3=\{\bar{0},\bar{1},\bar{2}\}$ and let $R=\{0,1\}\cup A$ be a set. We can apply the Construction S_3 or S_4 and the only H_v-field (here it is a hyperfield) which can be obtained is for $0\longmapsto\bar{0}$, $1\longmapsto\bar{1}$, $A\longmapsto\bar{2}$. Then the hyperoperations $\oplus$, $\odot$ defined in R are given by the relations $0\oplus 0=0$, $0\oplus 1=1$, $0\oplus a=A$, $1\oplus 1=A$, $1\oplus a=0$, $a\oplus b=1$ and $0\odot 0=0\odot 1=0\odot a=0$, $1\odot 1=1$, $1\odot a=A$, $a\odot b=1$ for all a,b in A, and both hyperoperations are commutative. Moreover, 0 and 1 are singles and so mutually scalars.

1.5.8 EXAMPLE

Let us consider the ring $(Z_2^v,+,\cdot)$. This ring, which is the expression of all finite Boolean rings, has every element multiplicativelly idempotent. According to Remark 1.5.6 one can take the greater hyperoperation suggested in the Construction S_4. Thus one can replace the elements which are different from the zero and the unit by any set each. The usefulness of this construction is that in direct sums of representations, the product of elements of each class give always the whole class. Note that 0 and 1 are single and so mutually scalars.

1.5.9 CONSTRUCTION S_5

Let $(R,\oplus,\odot)$ be a set with two hyperoperations which also can be partial hyperoperations. Take two elements $0,1\notin R$ and consider the set $R^* = R\cup\{0,1\}$. Consider any field (or ring), $(R^*,+,\cdot)$ defined on R^* with zero and unit elements respectively 0 and 1. Then we can take the H_b-field (or H_b-ring) $(R^*,\boxplus,*)$, where the hyperoperations are defined by putting $\boxplus=\oplus\cup+$ and $*=\odot\cup\cdot$. This type of H_v-field $(R^*,\boxplus,*)$ enjoys the properties: The 0 and 1 are scalar elements and every element $x\in R$ has a unique symmetric $-x$, and if $x\neq 0$, it has a unique inverse x^{-1}. However, 0,1 are not single.

1.5.10 S-CONSTRUCTION

Let $(S_i,\cdot), i\in I, |I|>1$ be a pairwise disjoint family of H_v-semigroups. Then in the set

$$S = \bigcup_{i\in I} S_i$$

a hyperoperation $\circledast$, called **S-hyperoperation**, is defined as follows:

$$x_i \circledast y_i = x_i y_i \quad \text{for all} \quad (x_i,y_i) \in S_i^2$$

$$x_i \circledast x_j = S_i \cup S_j \quad \text{for all } (x_i,x_j) \in S_i \times S_j,\ i\neq j.$$

Then the hyperstructure $(S,\circledast)$, called **S-construction**, is a H_v-group.

Proof

For any triple of S_i^3, $i\in I$ the WASS is valid since $(\cdot)$ is WASS. Moreover for all $(x_i,x_j,x_k) \in S_i\times S_j\times S_k$ where $i=j=k$ is not valid, it is obtained that

$$(x_i\circledast x_j)\circledast x_k = S_i \cup S_j \cup S_k = x_i\circledast(x_j\circledast x_k)$$

So $\circledast$ is WASS. Finally, since $|I|>1$, it is obtained for all $x_i\in S_i$

$$x_i\circledast S = \left(\bigcup_{j\in I-\{i\}} (S_i\cup S_j) \right) \cup x_i S_i = S,$$

likewise $S\circledast x_i = S$, thus $\circledast$ is reproductive. ∎

The S-construction can be applied for semihypergroups and in all cases $|S/\beta^*| = 1$.

On a given family of pairwise disjoint sets A_i one can consider the **total hyperoperation**: $ab=A_i$ for all $a,b\in A_i$, or one can use the least **incidence hyperoperation**: $ab=\{a,b\}$ for all $a,b\in A_i$, then the S-construction can be applied. The S-construction of hypergroups is essentially lexicographic.

Using the S-construction one can define H_v-groupoid algebra see paragraph 5.2.

1.5.11 DEFINITION

A H_v-structure is called **very thin** iff all hyperoperations are operations except one, which has all hyperproducts singletons except

only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hyperoperation $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $cardA > 1$, and all the other products, with respect to any other hyperoperation, are singletons.

For these hyperstructures see Appendix 2.

1.5.12 DEFINITION

Let H be an algebraic structure and take a subset $P\subset H$ for each operation $(\cdot)$. Then a **P-hyperoperation** P^* is defined by setting

$$xP^*y = xPy \quad \text{for all} \quad x,y\in H.$$

The set H with P-hyperoperations is called **P-hyperstructure**. For each associative operation the corresponding P-hyperoperation it is obtained to be also associative.

A slightly different definition can be obtained by setting

$$xP^*_{\wedge}y = xyP \quad \text{for all} \quad x,y\in H$$

and analogously for the left multiplication. However, some conditions, on the choose of P's, are needed in order the P-hyperoperations corresponding to associative operations to be WASS.

For an extensive study of this class see Appendix 3.

1.5.13 DEFINITION

Let $\langle H,*\rangle$ be a H_v-group, $(G,\cdot)$ be a group with identity 1, $\{A_i\}_{i\in G}$ be a family of non-empty sets indexed of G with $A_1=H$. Set $K=\bigcup_{i\in G} A_i$ and consider the hyperoperation $\circ$ defined in K as follows

$$x\circ y = x^*y \quad \text{for all} \quad x,y\in H$$

$$x\circ y = A_{ij} \quad \text{for all} \quad (x,y)\in A_i\times A_j \neq H\times H$$

Then $\langle K,\circ\rangle$ becomes H_v-group, which we call it **(H,G)-H_v-group**. It is easy to see that

$$K/\beta^* \cong G.$$

Analogous definition can be given for H_v-rings:

Let $\langle H,*,\circ\rangle$ be H_v-ring, where $\langle H,\circ\rangle$ is H_v-group, $(R,+,\cdot)$ be a ring with 0 the zero element, and $\{A_i\}_{i\in R}$ be a family of non-empty sets

indexed of R with $A_0=H$. Set $K=\bigcup_{i\in G} A_i$ and consider the hyperoperations $\oplus$, $\odot$ defined in K as follows

$$x\oplus y = x^*y \ , \quad x\odot y = x\circ y \quad \text{for all} \quad x,y\in H$$

$$x\oplus y = A_{i+j} \ , \quad x\odot y = A_{ij} \quad \text{for all} \quad (x,y)\in A_i\times A_j \neq H\times H$$

Then $\langle K,\oplus,\odot\rangle$ is H_v-ring, which we call it **(H,R)-H_v-ring**. One can also prove that

$$K/\gamma^* \cong R.$$

1.5.14 DEFINITION

A hyperstructure $\langle H,+\rangle$ will be called **H_v-canonical hypergroup** if $(+)$ is WASS and COW and

(1) There exists a $0\in H$ such that for all $x\in H$ $x+0=x$

(2) For all $x\in H$ there exists one and only one $-x\in H$ such that $0\in x+(-x)$ $(:x-x)$.

(3) $z\in x+y \Rightarrow y\in z-x$

If the associativity and the commutativity are valid then it is obtained the well known hyperstructure called **canonical hypergroup**.

This hyperstructure is the main Krasnerian type hyperstructure, see Appendix 1.

Τάξεις ιερουργών
δένουν την όρασή μας
πάνω απ' τη σκιά
που σκεπάζει το ρώτημά μας.

Classes of officiating priests
bind our vision
above the shadow
that covers our question.

CHAPTER 2

H_v - GROUPS

2.1 H_v-GROUPS DEFINED ON THE SAME SET

In a set H with n elements, one can define $(2^n-1)^{n^2}$ non-degenerate hyperoperations; so this is the number of hypergroupoids defined on H.

In a given hypergroupoid, it is laborious to check the WASS. In the following, some properties are presented to reduce the cases of the triples of elements for which one needs to check the WASS.

2.1.1 PROPOSITION

Let $(H,\cdot)$ be a COW hypergroupoid, then

$$a(ba) \cap (ab)a \neq \emptyset \quad \text{for all } a,b \text{ in } H.$$

Proof
Since $(H,\cdot)$ is COW, take $z \in ab \cap ba$. Then $za \cap az \neq \emptyset$, therefore, $(ab)a \cap a(ba) \neq \emptyset$. ∎

2.1.2 PROPOSITION

Let $(H,\cdot)$ be a commutative (strong) hypergroupoid. Then for all $a,b \in H$ we have:

$$a(ab) \cap (aa)b \neq \emptyset \quad \text{implies} \quad b(aa) \cap (ba)a \neq \emptyset.$$

Proof

Obvious, since $b(aa) = (aa)b$ and $(ba)a = a(ba) = a(ab)$ so $b(aa) \cap (ba)a \neq \emptyset$. ∎

2.1.3 REMARKS

From the above two propositions we obtain that the cases one have to check the WASS are:

(i) If $(H,\cdot)$ is COW, then for $a,b \in H$, with $a \neq b$, check the triples (a,a,b), (a,b,b), (b,a,a), (b,b,a) and every triple with three different elements.

(ii) If $(H,\cdot)$ is commutative (strong), then for $a,b \in H$, with $a \neq b$, check the triples (a,a,b), (b,b,a) and every triple with three different elements.

2.1.4 PROPOSITION

Let e be a unit of a hypergroupoid $(H,\cdot)$. If $x,y,z \in H$, such that $e \in \{x,y,z\}$ then

$$(xy)z \cap x(yz) \neq \emptyset.$$

Proof

Let $x = e$ then $(ey)z \supset yz$ and $e(yz) \supset yz$ so

$$(ey)z \cap e(yz) \neq \emptyset.$$

One checks the other cases in a similar way. ∎

ON THE SET $H = \{a,b\}$

In a set with two elements $H=\{a,b\}$, exactly eight hypergroups in the sense of Marty can be defined. For the H_v-groups the following is obtained:

2.1.5 THEOREM

All H_v-quasigroups with two elements are H_v-groups.

Proof
Besides the eight hypergroups, there are ten more H_v-quasigroups. It is easy to see that the WASS is also valid for them, so they are H_v-groups.

Therefore, the eighteen H_v-groups, up to isomorphism, defined on a set $H = \{a,b\}$, are given in the following table:

H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}	H_{16}	H_{17}	H_{18}
a	H	H	H	a	H	H	H	H	a	b	H	H	H	H	a	H	H
b	a	H	a	H	H	H	H	b	H	H	H	b	a	a	H	H	H
b	a	a	H	H	H	H	H	b	b	a	b	H	a	b	H	a	b
a	b	b	b	b	a	b	H	a	a	b	a	a	H	H	a	H	H

In this table in each column the H_v-groups $H_1, H_2, \ldots, H_{18}$ are presented. The hyperproducts $a\cdot a$, $a\cdot b$, $b\cdot a$, $b\cdot b$ are presented on the first, second, third and fourth row, respectively.

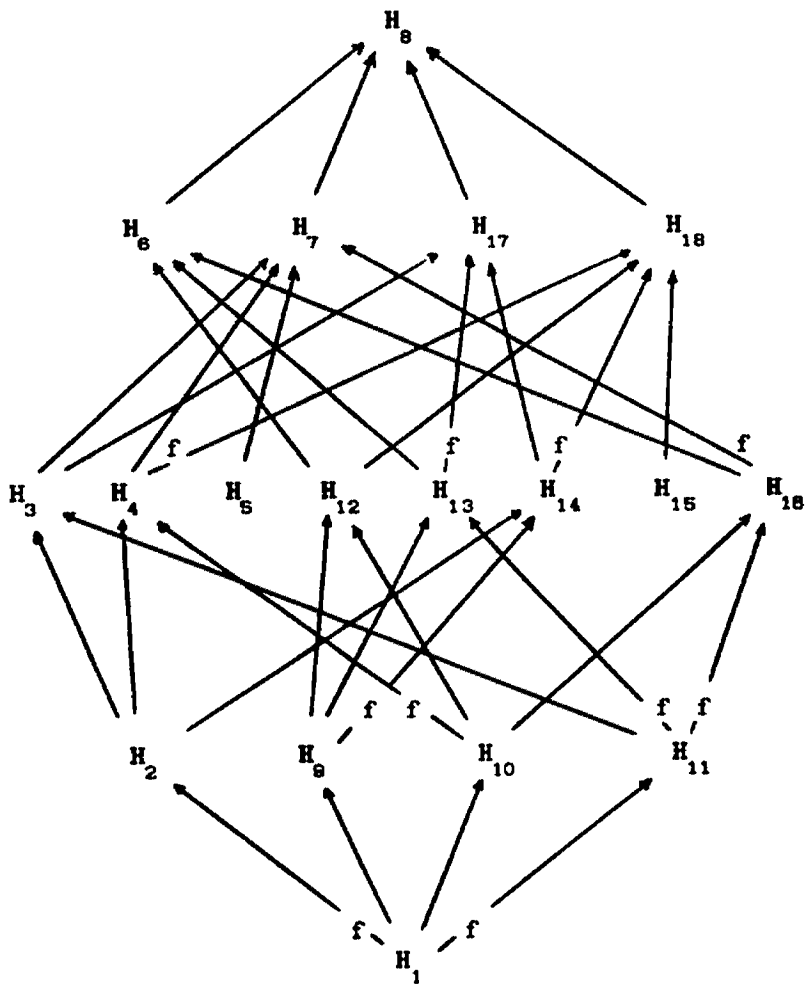
In the above table, the first eight H_v-groups are hypergroups. Moreover, all the above , except the H_{15}, are weak commutative. ■

Remark. The Theorem 2.1.5 is not true for $|H|\geq 3$. For example, if $H=\{a,b,c\}$, take the hyperoperation $(\cdot)$ defined as follows:

$aa=bb=cc=c$, $ab=ba=b$, $bc=cb=a$, and $ac=ca=H$.

Then $(H,\cdot)$ is a quasi-hypergroup but is not an H_v-group since $(ab)c=a$ and $a(bc)=c$.

2.1.6 REMARK
On the set of all H_v-groups over the set $H=\{a,b\}$, one can see that the defined partial order is given by the following diagram:



In this diagram, f means that the inclusion is taken using the only one non identity automorphism f: a $\longmapsto$ b, b $\longmapsto$ a.

Remark

Theorem 2.1.5 can be proved using Remark 2.1.6 and Theorem 1.1.6. One only has to verify the weak associativity for H_5 and H_{15}, since the remaining H_v-quasigroups have hyperoperations greater than H_1.

ON THE SET H = {e,a,b}

In this paragraph we find all H_v-groups with three elements which contain a scalar unit element e. Therefore it is determined the set of all H_v-groups defined on the set H={e,a,b} up to isomorphism. Notice that the number of all hypergroupoids (non degenerate) with a unit element, on a set with three elements, is

$(2^3-1)^4 = 2.401$.

2.1.7 PROBLEM

Find all H_v-groups with three elements which contain a scalar unit. Therefore, for H={e,a,b}, take the multiplication table

·	e	a	b
e	e	a	b
a	a	1	2
b	b	3	4

and then find all the fours, of non-empty subsets of H, in the place of the four (1,2,3,4), such that the reproduction axiom and the WASS is valid.

In the set H only one automorphism can be defined:

$$f: e \longleftrightarrow e, \quad a \longleftrightarrow b.$$

So two H_v-groups $(H,\cdot)$ and $(H,*)$ are isomorphic if:

$$a*a = f(bb), \quad a*b = f(ba), \quad b*a = f(ab) \quad \text{and} \quad b*b = f(aa).$$

Thus, the fours (aa, ab, ba, bb) and (f(bb), f(ba), f(ab), f(aa)) are isomorphic. In other words, instead of the table

1	2
3	4

, take the table

4	3
2	1

then replace a by b and b by a. The result is the isomorphic H_v-group to the first one.

2.1.8 THEOREM

Let $H=\{e,a,b\}$ be a set with three elements. Consider the set $\mathcal{U}$ of all quasi-hypergroups $(H,\cdot)$ defined in H, where e is the scalar unit. Then the subset of $\mathcal{U}$ with all not WASS quasi-hypergroups is described as follows:

Denote by a four **(aa, ab, ba, bb)** the only products needed to be defined. Then the fours which are **not** WASS, with the corresponding isomorphic to them fours, are the following 17:

(e, {e,b}, {e,b}, a)	≈	(b, {e,a}, {e,a}, e)
(e, {e,b}, H, a)	≈	(b, H, {e,a}, e)
(e, H, {e,b}, a)	≈	(b, {e,a}, H, e)
({e,b}, e, a, {e,a})	≈	({e,b}, b, e, {e,a})
({e,b}, e, a, H)	≈	(H, b, e, {e,a})
({e,b}, e, {a,b}, {e,a})	≈	({e,b}, {a,b}, e, {e,a})
({e,b}, e, {a,b}, H)	≈	(H, {a,b}, e, {e,a})
({e,b}, {e,b}, a, {e,a})	≈	({e,b}, b, {e,a}, {e,a})
({e,b}, {e,b}, a, H)	≈	(H, b, {e,a}, {e,a})
({e,b}, e, b, {e,a})	≈	({e,b}, a, e, {e,a})
({e,b}, e, b, H)	≈	(H, a, e, {e,a})
({e,b}, {e,a}, b, {e,a})	≈	({e,b}, a, {e,b}, {e,a})
({e,b}, a, b, {e,a})		
({e,b}, a, b, H)	≈	(H, a, b, {e,a})
({e,b}, a, {e,b}, H)	≈	(H, {e,a}, b, {e,a})
({e,b}, b, a, {e,a})		
({e,b}, b, a, H)	≈	(H, b, a, {e,a})

Proof

To prove this theorem the following algorithm was applied:

In the position 1 of the four (1,2,3,4) the sets {e}, {a}, {b}, {e,a}, {e,b}, {a,b} and H, were consequently put. Then for each case, the positions 2,3,4 were completed in order to obtain small hyperoperations such that the reproduction axiom is valid. In each case it was checked if the WASS was valid taking into account the Remarks 2.1.3. If the WASS was valid, then the result was considered a minimal H_v-group. If the WASS was not valid, then the

sets in the positions 2,3,4 were enlarged by adding new elements. Again the WASS was checked and this algorithm was continued. The procedure was stopped if the WASS was valid. In each result, whether it was WASS or not, the isomorphic, reproductive, hyperoperations were obtained and used to the next steps. In each enlargment of the hyperoperations it was checked if a WASS hyperoperation from the ones obtained previously, was already contained. The results are the ones written in the theorem.

Within the same procedure is obviously proved, the following, say "complement", theorem:

2.1.8 THEOREM

The set of all H_v-groups, with a scalar unit e, defined on a set with three elements $H=\{e,a,b\}$ is the following:

Every greater four (aa, ab, ba, bb) than the following ones, with the corresponding isomorphic ones, defines a H_v-group in H.

M_1	(e, b, b, {e,a})	≈	({e,b}, a, a, e)
M_2	(e, {e,b}, {e,b}, {a,b})	≈	({a,b}, {e,a}, {e,a}, e)
M_3	(e, H, H, a)	≈	(b, H, H, e)
M_4	(e, {a,b}, {a,b}, e)		
M_5	(e, H, H, b)	≈	(a, H, H, e)
M_6	(a, {e,b}, {e,b}, a)	≈	(b, {e,a}, {e,a}, b)
M_7	(a, H, H, b)		
M_8	(b, e, e, a)		
M_9	({e,b}, {e,a}, b, H)	≈	(H, a, {e,b}, {e,a})
M_{10}	({e,b}, b, {e,a}, H)	≈	(H, {e,b}, a, {e,a})

For a given H_v-group, not minimal, one can remove an element from one product xy, with $x,y\in H$, and then check if the remaining hyperproduct gives a H_v-group. So a series can be obtained, and if H is finite, leads to a minimal H_v-group.

Therefore, the above set of fours, is the set of minimal ones of all H_v-groups with three elements which have a scalar unit.

2.1.10 REMARKS

(i) One can classify all the minimals according to their fundamental group H/β^*. For this classification we obtain the following:

The H_v-group M_8 has $H/\beta^* = \{\{e\}, \{a\}, \{b\}\}$, so it has three elements in the fundamental group.

The H_v-group M_1 has $H/\beta^* = \{\{e,a\}, \{b\}\}$, so it has two elements in the fundamental groups.

The rest minimals have one element in the fundamental group.

(ii) The H_v-group M_1 is a very thin hypergroup. There is also the very thin H_v-group of the second kind. This has the four (e,b,b,H) and contains the M_1.

The other very thin H_v-groups can be obtained from the group M_8 by setting in one of the positions 1,2,3 or 4 greater subsets. These are H_b-groups and, under the isomorphism, we have the fours:

$(\{e,b\},e,e,a)$, $(\{a,b\},e,e,a)$, (H,e,e,a) ,
$(b,\{e,a\},e,a)$, $(b,\{e,b\},e,a)$, (b,H,e,a).

Therefore, in the set of all H_v-groups on $H=\{e,a,b\}$, there are exactly 8 very thin hypergroups.

(iii) From the minimals only the M_8 and M_{10} are not COW. Enlarging them, one can obtain only two more non COW in the set $H=\{e,a,b\}$. Therefore, in the set of all H_v-groups in H with e scalar unit, there are only four non COW H_v-groups: the M_8, M_{10} and $(H,\{e,a\},b,H)$, $(H,b,\{e,a\},H)$.

2.2 H_v-GROUPS AND H_b-GROUPS

2.2.1 THEOREM

Let $(H,\cdot)$ be H_v-group and A be a closed subset of H, i.e. $A\cdot A\subset A$. A necessary and sufficient condition in order A to be a H_v-subgroup is the following:

$$\left.\begin{array}{l} a(H-A)-(H-A)\subset aA \\ (H-A)a-(H-A)\subset Aa \end{array}\right\} \quad \text{for all } a\in A \qquad (I)$$

Proof
We prove this theorem for the left reproductivity, so we suppose that only the first condition in (I) is valid and the rest of the proof is similar.
Suppose there exists $b\in A-aA$ for some $a\in A$. Then there exists an element $x\in H$ such that $b\in ax$.

(a) If $x\in A$ then $b\in ax\subset A$, contradiction.

(b) If $x\in H-A$ then we have

$b\in\{b\}-(H-A)\subset ax-(H-A)\subset a(H-A)-(H-A)\subset aA$, contradiction.

Therefore $A-aA=\emptyset$, so for all $a\in A$ we have $A\subset aA$ and the left reproductivity is valid.

Now let us suppose that the left reproductivity is valid in A, then $aA=A$ for all $a\in A$, thus $a(H-A)-(H-A)\subset A=aA$. So (I) is valid. ∎

2.2.2 TRHEOREM

Let $(H,\cdot)$ be a finite H_v-group. For every element $a\in H$ there exists a power, take the minimal one, a^{λ} that contains an element of one of the previous powers, i. e. there exists a^{κ} such that $a^{\kappa}\cap a^{\lambda}\neq\emptyset$, $\kappa<\lambda$. Then $a^{\lambda-\kappa}\subset\omega_H$.

Proof
From $a^{\kappa}\cap a^{\lambda}\neq\emptyset$ we have $\varphi(a^{\kappa})=\varphi(a^{\lambda})$ or $(\varphi(a))^{\kappa}=(\varphi(a))^{\lambda}$, where φ is the fundamental map.

But $(\varphi(a))^{\kappa}$ and $(\varphi(a))^{\lambda}$ are elements of a group, so $(\varphi(a))^{\lambda-\kappa}=\omega_H$ because ω_H is the identity of H/β^*. Therefore, $\varphi(a^{\lambda-\kappa})=\omega_H$, so $a^{\lambda-\kappa}\subset\omega_H$. ∎

2.2.3 REMARKS

(i) In the infinite case if there are powers κ,λ , with $\lambda>\kappa$, such that $a^{\kappa} \cap a^{\lambda} \neq \emptyset$ then again we have $\varphi(a^{\lambda-\kappa}) = \omega_H$ so $a^{\lambda-\kappa} \subset \omega_H$.

(ii) If for some $\mu<\lambda-\kappa$ the relation $a^{\kappa+\mu} \cap a^{\lambda} \neq \emptyset$ is also valid, then we obtain $a^{\lambda-\kappa-\mu} \subset \omega_H$. So there exists lower power, of the element a, than the $\lambda-\kappa$ contained in the core.
If two succesive powers a^{μ}, $a^{\mu+1}$ have a common element then $a \in \omega_H$, therefore, all the powers of the element a belong to the core.

(iii) Let A be a closed set of an H_v-group, i.e. $A \cdot A \subset A$. If $\varphi(A)$ is finite then $A \cap \omega_H \neq \emptyset$.
Indeed, from $\varphi(A)\varphi(A) \subset \varphi(A)$ it is obtained that $\varphi(A)$ is a closed finite subset of the quotient group H/β^*. Therefore, $\varphi(A)$ is a subgroup, so $\omega_H \in \varphi(A)$. Thus $A \cap \omega_H \neq \emptyset$.

(iv) If $a^{\kappa} \cap a^{\lambda} \neq \emptyset$ then for every natural number μ it is obtained $a^{\kappa+\mu}\ \beta^*\ a^{\lambda+\mu}$. Indeed,
$a^{\kappa} \cap a^{\lambda} \neq \emptyset$ implies $\varphi(a^{\kappa}) = \varphi(a^{\lambda})$ so $\varphi(a^{\kappa+\mu}) = \varphi(a^{\lambda+\mu})$.

2.2.4 REMARK

If G is a sub-H_v-group of the H_v-group H, then

$$\beta_G^*(x) \subset \beta_H^*(x) \quad \text{for all } x \in G$$

where $\beta_G^*(x)$, $\beta_H^*(x)$ are the fundamental classes of x with respect to the H_v-groups G and H, respectively. The same relation is valid for the γ^* and ε^* relations.

However, the relation $|G/\beta^*| \leq |H/\beta^*|$ does not necessarily valid.

Example: Consider the example of the feebly canonical hypergroups given in [4]. Let G be an abelian group and S a set, such that $|G|>1<|S|$ and $G \cap S=\emptyset$. In $G \cup S$ a hyperproduct $(\circ)$ is defined as follows:

$$\left.\begin{array}{ll} g\circ s = s\circ g = S & \text{for all } (g,s)\in G\times S \\ g_1\circ g_2 = g_1g_2 & \text{for all } (g_1,g_2)\in G^2 \\ s_1\circ s_2 = G & \text{for all } (s_1,s_2)\in S^2 \end{array}\right\}$$

then $\langle G\cup S,\circ\rangle$ becomes a feebly canonical hypergroup.

For $|G|=n>2$ it is obtained

$$n = |G/\beta^*| > |(G\cup S)/\beta^*| = 2.$$

A **polygroup** [2] is a system $\mathcal{M} = \langle M,\cdot,e,^{-1}\rangle$ where $e\in M$, $^{-1}$ is a unitary operation on M, $(\cdot)$ maps M^2 into nonempty subsets of M, and the following axioms hold for all x,y,z in M :

$$(x\cdot y)\cdot z = x\cdot(y\cdot z)\ ,\quad e\cdot x = x = x\cdot e\ ,\quad \text{and}$$
$$x\in y\cdot z \quad \text{implies} \quad y\in x\cdot z^{-1} \quad \text{and} \quad z\in y^{-1}x.$$

An extension of polygroups by polygroups have been introduced in the following way:

Suppose $\mathcal{U}=\langle A,\cdot,e,^{-1}\rangle$ and $\mathcal{B}=\langle B,\cdot,e,^{-1}\rangle$ are polygroups whose elements have been renamed so that $A\cap B=\{e\}$ where e is the identity of both $\mathcal{U}$ and $\mathcal{B}$. A new system $\mathcal{U}[\mathcal{B}] = \langle M,*,e,^{I}\rangle$, called the extension of $\mathcal{U}$ by $\mathcal{B}$, is formed in the following way :

Set $M = \{x\in A: x\neq e\}\cup\{x\in B: x\neq e\}\cup\{e\}$ and let $e^I=e$, $x^I=x^{-1}$, $e*x=x*e=x$ for all $x\in M$, and for all $x,y\in\{x\in M : x\neq e\}$,

$$x*y = \begin{cases} x\cdot y & \text{if } x,y\in A \\ x & \text{if } x\in B,\ y\in A \\ y & \text{if } x\in A,\ y\in B \\ x\cdot y & \text{if } x,y\in B,\ y\neq x^{-1} \\ x\cdot y\cup A & \text{if } x,y\in B \text{ and } y=x^{-1} \end{cases}$$

The extension $\mathcal{U}[\mathcal{B}]$ is a polygroup which preserves being chromatic.

2.2.5 **THEOREM**

$$\mathcal{U}[\mathcal{B}]/\beta^* \cong \mathcal{B}/\beta^*$$

Proof

We have by definition: $x*x^{-1} = x\cdot x^{-1}\cup A$ therefore, all the elements of A belong to the fundamental class $\beta^*_{\mathcal{U}[\mathcal{B}]}(e)$. Moreover for the elements of B we have $b\beta^*_{\mathcal{U}[\mathcal{B}]}b_1$ iff $b\beta^*_{\mathcal{B}}b_1$; that is because

two elements of B can be contained only in the hyperproducts of elements of B.

Therefore, the extension $\mathcal{U}[\mathcal{B}]$ has in the unit fundamental class all the elements of A and the other classes have exactly the elements of $\mathcal{B}/\beta^*_{\mathcal{B}}$. ∎

The next theorem is referred to the P-H_v-groups, so one can see for the notation in Appendix 3. Note that in the following $1 \in G$ so a P-H_b-group is obtained.

2.2.6 THEOREM

Let Z be the center of a group $(G,\cdot)$, then

$$<G,Z^*>/\beta^* \cong G/Z$$

Proof

Let us denote by $^*\prod$ the hyperproduct in $<G,Z^*>$ and by $\cdot\prod$ the product in $(G,\cdot)$. Then

$a\beta^*b \longleftrightarrow$ there exist x_i, h_{i_j} $i \in \{1,\dots,k\}$ such that

$$\{a,x_1\} \subset {}^*\prod_{i_1 \in I_1} h_{i_1} ,\dots, \{x_{k-1},b\} \subset {}^*\prod_{i_k \in I_k} h_{i_k} \longleftrightarrow$$

$$\{a,x_1\} \subset \cdot\prod_{i_1 \in I_1} h_{i_1} Z ,\dots, \{x_{k-1},b\} \subset \cdot\prod_{i_k \in I_k} h_{i_k} Z \longleftrightarrow$$

$$a=h_{a,x_1}z_a,\ x_1=h_{a,x_1}z_{x_1},\dots,x_{k-1}=h_{x_{k-1},b}\cdot z_{x_{k-1}},\ b=h_{x_{k-1},b}\cdot z_b \longleftrightarrow$$

$$h_{a,x_1}=az_a^{-1}=x_1z_{x_1}^{-1},\dots,\ h_{x_{k-1},b}=x_{k-1}z_{x_{k-1}}^{-1}=bz_b^{-1} \longleftrightarrow$$

there are $z_{a,x_1},\dots,z_{x_{k-1},b} \in Z$ such that

$$a=x_1z_{a,x_1},\dots,\ x_{k-1}=bz_{x_{k-1},b} \longleftrightarrow a=bz_{x_{k-1},b}\cdots z_{a,x_1} \longleftrightarrow$$

$a = bz \cdot \longleftrightarrow a \equiv b \bmod Z$. ∎

2.3 TRANSLATIONS

Using left multiplications by an element of a H_v-group a new H_v-group can be defined.

2.3.1 DEFINITION

Let $(H,\cdot)$ be a H_v-group and $x\in H$. Then the ℓ_x-**fundamental relation** is defined as follows:

$$a\ \ell_x\ b \quad \text{iff} \quad \exists z_1,\dots,z_{k-1}\ ;\ h_1,\dots,h_k \in H \quad \text{such that}$$

$$\{a,z_1\} \subset xh_1\ ,\ \{z_1,z_2\} \subset xh_2\ ,\dots,\ \{z_{k-1},b\} \subset xh_k$$

This is an equivalence relation but the product of two classes $\ell_x(a)$ and $\ell_x(b)$ is defined in the usual manner and it is a hyperoperation

$$\ell_x(a)\circ\ell_x(b) = \{\ \ell_x(c) : c \in \ell_x(a)\cdot\ell_x(b)\ \}\ .$$

The $(\ H/\ell_x,\circ\ )$ is a H_v-group called **left fundamental hypergroup** corresponding to the element x. There are as many left fundamental H_v-groups as the elements of H. Analogously, the right fundamental hypergroups are defined. For every $x,h\in H$ we have $\ell_x(h) \subset \beta^*(h)$.

2.3.2 THEOREM

For every H_v-group $(H,\cdot)$ and for every $x\in H$ we have

$$H/\beta^* \ \cong\ (H/\ell_x)/\beta^*$$

Proof
Consider the map

$$f : H/\beta^* \longrightarrow (H/\ell_x)/\beta^* : \beta^*(h) \longmapsto \beta^*(\ell_x(h))$$

For every $h_1,h_2\in H$ and $z\in h_1h_2$ it is obtained

$$f(\beta^*(h_1)\cdot\beta^*(h_2)) = f(\beta^*(h_1h_2)) = f(\beta^*(z)) = \beta^*(\ell_x(z))\ .$$

On the other hand

$$\beta^*(\ell_x(h_1))\cdot\ \beta^*(\ell_x(h_2)) = \beta^*(\ell_x(h_1)\cdot\ell_x(h_2)) = \beta^*(Z)$$

for every $Z \in \ell_x(h_1)\cdot\ell_x(h_2)$.

Therefore, f is a homomorphism.

Now let $\beta^*(h_1) \neq \beta^*(h_2)$ and let us suppose that

$$\beta^*(\ell_x(h_1)) = \beta^*(\ell_x(h_2)) .$$

then

$$\{\ell_x(h_1),\ell_x(z_1)\} \subset \prod_{i_1\in I_1} \ell_x(k_{i_1}),\ldots,\{\ell_x(z_{j-1}),\ell_x(h_2)\} \subset \prod_{i_j\in I_j} \ell_x(k_{i_j})$$

From the first inclusion it is obtained

$$\{\ell_x(h_1),\ell_x(z_1)\} \subset \{\ell_x(c): c \in \prod_{i_1\in I_1} \ell_x(k_{i_1})\} \subset \{\ell_x(c): c \in \prod_{i_1\in I_1} \beta^*(k_{i_1})\}$$

But the product of fundamental classes is singleton therefore

$$\{ \ell_x(h_1),\ \ell_x(z_1) \} \subset \{ \ell_x(c) : c \in \beta^*(w) \} ,$$

where w is any element of $\prod_{i_1\in I_1} \beta^*(k_{i_1})$, so $\beta^*(h_1) = \beta^*(z_1)$.

Similarly, $\beta^*(z_1)=\beta^*(z_2)$,...., $\beta^*(z_{j-1})=\beta^*(h_2)$, so $\beta^*(h_1)=\beta^*(h_2)$

which is a contradiction.

Therefore f is an isomorphism. ∎

2.3.3 THEOREM

Let $(H,\cdot)$ be H_v-group and let L_a be the left translation by a, then the following are valid

(i) $L_a(\beta^*(x)) \subset \beta^*(L_a(x))$ for all x in H

(ii) If H is finite, then for all a in H the L_a is an inner automorphism (permutation) of the fundamental group H/β^*. Moreover, if a is left scalar, then L_a preserves the cardinality of the corresponding classes:

$$|L_a(\beta^*(x))| = |\beta^*(x)|$$

Proof

(i) If $\{x,y\} \subset \prod h_i$, $h_i \in H$, then $L_a\{x,y\} \subset L_a\prod h_i$ therefore,

$(ax)\cup(ay) \subset a\prod h_i$. Thus for the transitive closure it is obtained that $(ax)\beta^*(ay)$ which proves the (i).

(ii) The first part is obvious from (i) and the reproductivity in H/β^*. Now if a is left scalar, then for $x\neq y$ we have $ax\neq ay$ otherwise the reproductivity is not valid, so

$$|\beta^*(x)| \leq |L_a\beta^*(x))|.$$

On the other side from the reproductivity of classes we have

$$|L_a(\beta^*(x))| \leq |\beta^*(x)|,$$

therefore the theorem is proved. Analogous theorem is valid for right scalars. ■

Remark

If a is a left scalar then the above Theorem establishes a one to one correspondance between the elements of the corresponding classes. Thus one can have some results as the following one: If there exists a $x\in H$ which has $|\beta^*(x)|=r$, and there is no other class with cardinality r then $a\in\omega_H$. This is so since we must have $|a\beta^*(x)| = |\beta^*(x)|$.

2.4 SEMIDIRECT HYPERPRODUCT

Using the fundamental equivalence relation one can define semidirect hyperproducts of H_v-groups, cf [32].

2.4.1 THEOREM

Let A,B be H_v-groups and let's take the group $AutA$. Consider any homomorphism

$$\wedge : B/\beta^* \longrightarrow AutA : \beta^*(b) \longmapsto \beta^*(b) \xrightarrow{\text{denote}} \hat{b}$$

then in $A\times B$ a hyperproduct can be defined by the relation

$$(a,b)(a_1,b_1) = \{ (x,y) : x \in a\hat{b}(a_1),\ y \in bb_1 \} = (a\hat{b}(a_1), bb_1)$$

The set $A\times B$ becomes a H_v-group called **semidirect hyperproduct** of A and B corresponding to $(\wedge)$ and it is denoted by $A\hat{\times}B$.

Proof
From the fundamental property, for all $x,y\in B$ it is valid

$$x\hat{\cdot}y = \hat{x}\circ\hat{y} = \hat{z} \qquad \text{for all } z\in x\cdot y.$$

Now, let $(a,b), (a_1,b_1), (a_2,b_2) \in A\times B$ then

$$((a,b)(a_1,b_1))(a_2,b_2) = \bigcup_{\substack{x\in a\hat{b}(a_1)\\ y\in bb_1}} (x,y)(a_2,b_2) =$$

$$= \bigcup_{\substack{x\in a\hat{b}(a_1)\\ y\in bb_1}} \{ (z,w) \mid z\in x\hat{y}(a_2), w\in yb_2 \} =$$

$$= \{(z,w) \mid z\in (a\hat{b}(a_1))b\hat{\cdot}b_1(a_2), w\in (bb_1)b_2\} = ((a\hat{b}(a_1))b\hat{\cdot}b_1(a_2), (bb_1)b_2)$$

On the other hand,

$$(a,b)((a_1,b_1)(a_2,b_2)) = \bigcup_{\substack{x\in a_1\hat{b}_1(a_2)\\ y\in b_1b_2}} (a,b)(x,y) =$$

$$= \bigcup_{\substack{x\in a_1\hat{b}_1(a_2)\\ y\in b_1b_2}} \{ (z,w) \mid z\in a\hat{b}(x), w\in by \} =$$

$$= \{ (z,w) \mid z\in a\hat{b}(a_1\hat{b}_1(a_2)),\ w\in b(b_1b_2) \} =$$

$$= \{ (z,w) \mid z\in a(\hat{b}(a_1)\hat{b}(\hat{b}_1(a_2))),\ w\in b(b_1b_2) \} =$$

$$= \{ (z,w) \mid z\in a(\hat{b}(a_1)\hat{b}\circ\hat{b}_1(a_2)),\ w\in b(b_1b_2) \} =$$

$$= \{ (z,w) \mid z\in a(\hat{b}(a_1)b\hat{\cdot}b_1(a_2)),\ w\in b(b_1b_2) \} =$$

$$= (a(\hat{b}(a_1)b\hat{\cdot}b_1(a_2)),\ b(b_1b_2)).$$

But A,B are H_v-groups, so

$$(a\hat{b}(a_1))b\hat{\cdot}b_1(a_2) \cap a(\hat{b}(a_1)b\hat{\cdot}b_1(a_2)) \neq \emptyset \quad \text{and} \quad (bb_1)b_2 \cap b(b_1b_2) \neq \emptyset.$$

Therefore it yields

$$((a,b)(a_1,b_1))(a_2,b_2) \cap (a,b)((a_1,b_1)(a_2,b_2)) \neq \emptyset$$

i.e. the WASS is valid.

It is also easy to see that the reproductivity is valid. Thus the semidirect hyperproduct is H_v-group. ∎

REMARK

In the case that in A there exists an absolute unit element e_a, then for every $b \in B$ for which $\hat{b}(e_a)=e_a$ it is obtained $\{e_a\}\hat{\times}B \cong B$. Similarly, if e_b is any unit element of B such that $e_b e_b = e_b$, then $A\hat{\times}\{e_b\} \cong A$. This remark can be applied, for example, for reversible hypergroups, canonical hypergroups and polygroups.

In $A \times B$ we consider an equivalence relation $\mathcal{A}$ as follows:

$(a,b) \mathcal{A} (a_1,b_1)$ iff $b = b_1$.

Then it is easy to see that the following theorem is valid.

2.4.2 THEOREM

$$A \hat{\times} B / \mathcal{A} \cong B$$

Therefore, $A\hat{\times}B$ is said to be an extension of the H_v-group A by the H_v-group B. For another extension of a group by hypergroup see [27]. Moreover, in [2] there is an extension of polygroups by polygroups defined by "enlarging" the unit element that exists in polygroups.

2.4.3 LEMMA

Let A,B be H_v-groups and take $\hat{} : B/\beta^* \longrightarrow \mathrm{Aut}A$. Then

$$\hat{b}(\beta^*(a)) = \beta^*(\hat{b}(a)) \quad \text{for all } a \in A, b \in B.$$

Proof

Let $x \in \hat{b}(\beta^*(a))$; then there exists $c \in \beta^*(a)$ such that $x=\hat{b}(c)$. So there are $h_{i_j} \in A$, $i_j \in I_j$ such that

$$\{c,z_1\} \subset \prod_{i_1 \in I_1} h_{i_1}, \ldots, \{z_{\lambda-1},a\} \subset \prod_{i_\lambda \in I_\lambda} h_{i_\lambda}$$

which implies

$$\{\hat{\delta}(c),\hat{\delta}(z_1)\} \subset \prod_{i_1\in I_1} \hat{\delta}(h_{i_1}) , \ldots , \{\hat{\delta}(z_{\lambda-1}),\hat{\delta}(a)\} \subset \prod_{i_\lambda\in I_\lambda} \hat{\delta}(h_{i_\lambda})$$

i.e. $\hat{\delta}(c)\beta^*\hat{\delta}(a)$, thus $x\in\beta^*(\hat{\delta}(a))$.

Conversely, let $x\in\beta^*(\hat{\delta}(a))$ or $x\beta^*\hat{\delta}(a)$. Then we can have

$$\{x,z_1\} \subset \prod_{i_1\in I_1} h_{i_1} , \ldots , \{z_{\lambda-1},\hat{\delta}(a)\} \subset \prod_{i_\lambda\in I_\lambda} h_{i_\lambda}$$

If the inverse automorphism $\hat{\delta}^{-1}$ of $\hat{\delta}$ is applied, it is obtained

$$\{\hat{\delta}^{-1}(x),\hat{\delta}^{-1}(z_1)\} \subset \prod_{i_1\in I_1} \hat{\delta}^{-1}(h_{i_1}),\ldots, \{\hat{\delta}^{-1}(z_{\lambda-1}),a\} \subset \prod_{i_\lambda\in I_\lambda} \hat{\delta}^{-1}(h_{i_\lambda})$$

so $\hat{\delta}^{-1}(x)\beta^*a$ or $\hat{\delta}^{-1}(x)\in\beta^*(a)$, therefore, $x\in\hat{\delta}(\beta^*(a))$. ∎

2.4.4 LEMMA

Let A,B be H_v-groups and consider a semidirect hyperproduct defined by the $(\wedge)$. Take $a,a_1\in A$; $b,b_1\in B$, then

$$\{(a,b),(a_1,b_1)\} \subset (h_1,k_1)\ldots(h_v,k_v) \quad \text{for some} \quad h_1\in A,\ k_1\in B$$

if and only if

$$\{a,a_1\} \subset (\ldots(h_1'h_2')\ldots)h_v' \quad \text{and} \quad \{b,b_1\} \subset (\ldots(k_1k_2)\ldots)k_v$$

for some $h_1'\in A$, $k_1\in B$.

Proof

Observe that

$$(h_1,k_1)\cdots(h_v,k_v) =$$

$$\left[\left[\ldots(h_1\hat{k}_1(h_2))\cdot k_1\hat{k}_2(h_3)\ldots\right]\overbrace{(\ldots(k_1k_2)\ldots)k_{v-1}}(h_1),\left[\ldots(k_1k_2)\ldots\right]k_v\right]$$

Therefore the " $\longrightarrow$ " is clear.

The " $\longleftarrow$ " is also clear because for given $h_1'\in A$ such that

$$\{a,a_1\} \subset (\ldots(h_1'h_2')\ldots)h_v'$$

one can take the elements

$$h_1 = h'_1 \ , \ h_2 = \hat{k}_1^{-1}(h'_2), \ \ldots \ , \ h_v = \overbrace{(\ldots(k_1k_2)\ldots)k_{v-1}}^{\wedge}{}^{-1}(h_1) \qquad \blacksquare$$

2.4.5 THEOREM

Let A,B be H_v-groups and $A\hat{\times}B$ a semidirect hyperproduct defined by the homomorphism $\hat{}: B/\beta^* \longrightarrow \mathrm{Aut}A$. Then $\hat{}$ induces a homomorphism from B/β^* to A/β^*. We have

$$A\hat{\times}B/\beta^* \ \cong \ A/\beta^* \ \hat{\times} \ B/\beta^*$$

Proof

The map

$$\varphi \ : \ \beta^*(a,b) \longmapsto (\beta^*(a), \beta^*(b))$$

is a homomorphism because, using Lemma 2.4.3, we have

$$\varphi(\beta^*(a,b)\cdot\beta^*(a_1,b_1)) = \varphi(\beta^*((a,b)(a_1,b_1))) =$$

$$= \varphi(\beta^*(a\hat{b}(a_1), bb_1)) = (\beta^*(a\hat{b}(a_1)), \ \beta^*(bb_1)) =$$

$$= (\beta^*(a)\cdot\beta^*(\hat{b}(a_1)), \beta^*(b)\beta^*(b_1)) = (\beta^*(a)\cdot\hat{b}(\beta^*(a_1)), \beta^*(b)\beta^*(b_1)) =$$

$$= (\beta^*(a), \beta^*(b))(\beta^*(a_1), \beta^*(b_1)) = \varphi(\beta^*(a,b))\cdot\varphi(\beta^*(a_1,b_1)) \ .$$

Finally, using Lemma 2.4.4, we have

$$(a,b) \ \beta^*(a_1,b_1) \qquad \text{iff}$$

$$\{ \ (a,b),(x_1,y_1) \ \} \subset \prod_{i_1 \in I_1} (h_{i_1}, k_{i_1}), \ldots,$$

$$\{ \ (x_{\lambda-1}, y_{\lambda-1}), (a_1,b_1) \ \} \subset \prod_{i_\lambda \in I_\lambda} (h_{i_\lambda}, k_{i_\lambda})$$

$$\text{iff} \qquad a\beta^*a_1 \quad \text{and} \quad b\beta^*b_1 \ .$$

2.5 CYCLICITY

Let $(H,\cdot)$ be H_v-group, then the powers of an element $x\in H$, with respect to the s-ary circle hyperoperation $\circledast$, are the following

$$x^1=\{x\},\ x^2=x\circledast x,\ \ldots,\ x^s=x\circledast\ldots\circledast x,\ \ldots$$

Given a subset $G\subset H$, denote by $[G]$ the H_v-semigroup generating by G, i.e. the set of all elements of H which are contained to a product of elements of G. In the special case where $S=\{x\}$, then

$$[x]=x^1\cup x^2\cup\ldots\cup x^s\cup\ldots$$

Fix a series of patterns of parentheses $p_1,\ldots,p_s,\ldots$ such that $p_i\in P_i$, for example the series of perentheses on the left or the series of parentheses on the right. Then the definitions of powers and of $[x]$ can be given as well.

In the following the powers are considered with respect to the circle hyperoperation, unless otherwise stated.

2.5.1 DEFINITIONS

The H_v-group $(H,\cdot)$ will be called **cyclic with finite period** with respect to $h\in H$, if there exists a positive integer $s\in Z^+$, such that

$$H=h^1\cup h^2\cup\ldots\cup h^s.$$

The minimum such a s will be called **period** of the **generator h**. If all generators have the same period, then H is **cyclic with period**. If there exists $h\in H$ and $s\in Z^+$, the minimum one, such that $H=h^s$ then H will be called **single-power cyclic** and h is a generator with **single-power period** s, cf [30].

The cyclicity in the infinite case is defined similarly. Thus, for example, the H_v-group $(H,\cdot)$ is called **single-power cyclic with infinite period** with generator h if every element of H belongs to a power of h and there exists $s_o\geq 1$, such that for every $s\geq s_o$ we have

$$h^1\cup h^2\cup\ldots\cup h^{s-1}\subsetneq h^s.$$

The above definitions are given for any other fixed series of patterns of parentheses.

2.5.2 REMARKS

(i) Consider the powers with respect to the series of parentheses on the left (respectively on the right) then, if for some s the relation

$$x^{s+1} \subset x^1 \cup x^2 \cup \ldots \cup x^s \qquad (1)$$

is valid, then

$$[x] = x^1 \cup x^2 \cup \ldots \cup x^s \qquad (2)$$

Indeed, multiplying the (1) by x on the right, it is obtained

$$x^{s+2} \subset x^2 \cup x^3 \cup \ldots \cup x^{s+1} \subset x^1 \cup x^2 \cup \ldots \cup x^s.$$

This result is obtained if the associativity is valid, where there is no difference between the series of patterns on perentheses.

Note that if the H_v-group is of finite order n, then the minimal number s for which (1) is valid is $s \leq n$.

(ii) If a H_v-group is single-power cyclic of single-power period s_1 of the generator h, then obviously it is cyclic but $s \leq s_1$, where s be the period of h. Moreover, even in the case where the H_v-group is finite, we may have $n < s_1$. For example, consider the fourth hypergroup in the Theorem 2.1.5, with two elements. Then the element b is a generator of singe-power period 3.

(iii) If all the elements of a H_v-group are single power generators then H is called H_v-group "renaissant".

(iv) The subset [x] is not necessarily a sub-H_v-group even in the case where the associativity is valid. Therefore, an open problem is to find a necessary and sufficient condition such that [x] is a sub-H_v-group.

2.5.3 THEOREM

Let $(G, \cdot)$ be a group with $|G|=n$ and for $P \neq \emptyset$, $P \subset G$, consider the P-hypergroup $\langle G, P^* \rangle$. Then, for all $x \in G$ the set [x] is a P-subhypergroup.

Proof
Let $a=xp_1x\cdots xp_rx$ and $b=xq_1x\cdots xq_sx$ be two elements of the r+1 and s+1 powers of x respectively with respect to P^*. Consider the positive integer $t\leq n$ such that $(p_1x\cdots xp_rx)^t= e$ where e is the unit element of the group G.

Take the element $y\in[x]$ such that

$$y = (xp_2x\cdots xp_rx)(p_1xp_2x\cdots xp_rx)^{t-2}\cdot q_1xq_2x\cdots xq_sx$$

Observe that

$$aP^*y = xp_1x\cdots xp_rx\cdot P\cdot(xp_2x\cdots xp_rx)(p_1xp_2x\cdots xp_rx)^{t-2}\cdot q_1xq_2x\cdots xq_sx \ni$$

$$\ni xp_1x\cdots xp_rx\cdot p_1\cdot xp_2x\cdots xp_rx(p_1xp_2x\cdots xp_rx)^{t-2}\cdot q_1xq_2x\cdots xq_sx =$$

$$= x(p_1xp_2x\cdots xp_rx)^t\cdot q_1xq_2x\cdots xq_sx = xq_1xq_2x\cdots xq_sx = b.$$

Therefore, for every $a,b\in[x]$ there exists a $y\in[x]$ such that $b \in aP^*y$.

Similarly, there exists a $z\in[x]$ such that $b \in zP^*a$. Thus, $[x]$ is a subhypergroup of $\langle G,P^*\rangle$. ∎

For more results on cyclic P-hypergroups see Appendix 3.

Μεγένθυνε το χάρτη
να δούμε
την ύπαρξή μας, τον προορισμό μας.

Blow up the map
so we can see
our existence, our destination.

CHAPTER 3

H_v - RINGS

3.1 FUNDAMENTAL RELATIONS

Let $(R,+,\cdot)$ be H_v-ring. If only the $(+)$ is a hyperoperation then R is called **additive H_v-ring**. If only the $(\cdot)$ is a hyperoperation then R is called **multiplicative H_v-ring**.

Left (respectivelly right) H_v-ideal in a H_v-ring $(R,+,\cdot)$ is called a subset $I \subset R$, where $(I,+)$ is H_v-group and $IR \subset I$ (resp. $RI \subset I$).

In what follows we focus our attention on the β^*, γ^* relations defined on H_v-rings. Notice that two kinds β^* relations can be defined on H_v-rings. We denote by β^*_+, $\beta^*_\cdot$ the β^* relation with respect to addition and to multiplication, respectively.

3.1.1 DEFINITION

We define the γ_1^* , γ_2^* relations as the transitive closure of the relations γ_1, γ_2, respectively, which are defined as follows:

$x\gamma_1 y$ iff $\exists\, a_i \in R$ and I_k, K finite sets of indices such that

$$\{x,y\} \subset \sum_{k\in K} \left(\prod_{i\in I_k} a_i \right)$$

and

$x\gamma_2 y$ iff $\exists\, b_j \in R$ and J_s, S finite sets of indices such that

$$\{x,y\} \subset \prod_{s\in S} \left(\sum_{j\in J_s} b_j \right)$$

Notice that in hyperrings we have $\gamma_1 = \gamma$.

3.1.2 PROPOSITION

(i) R/γ_1^* is a multiplicative H_v-ring.

(ii) R/γ_2^* is an additive H_v-ring.

Proof

We prove (i); (ii) can be proved in a similar way. The sum of the classes is

$$\gamma_1^*(x) \oplus \gamma_1^*(y) = \{\gamma_1^*(z) : z \in \gamma_1^*(x) + \gamma_1^*(y)\}$$

In the definition of γ_1^* the expressions $v = \sum (\prod a)$ are used. In the above sum of classes the element z belongs to the sums of such expressions v, therefore z also belongs to a sum of products. That means that all z's are in the same γ_1^* class. So, the sum of γ_1^*-classes is a singleton. Therefore, R/γ_1^* is a multiplicative H_v-ring. ∎

Remark that the γ_1^* classes are greater than the β^* classes. Actually, the γ_1^* is not the smallest equivalence relation such that $\langle R/\gamma_1^*, \oplus\rangle$ to be a group. To see this, take a multiplicative H_v-ring R. Then $R/\beta^* \cong R$, but R/γ_1^* is not isomorphic to R.

3.1.3 PROPOSITION

For all additive H_v-rings we have $\gamma_1^* = \beta_+^*$

For all multiplicative H_v-rings we have $\gamma_2^* = \beta_\cdot^*$

Proof

The proof is similar in both parts. We give the proof for the multiplicative H_v-ring R. In this case every sum of elements of R is a singleton. Therefore,

$$\prod_{s\in S}\left(\sum_{j\in J_s} b_j\right) = \prod_{s\in S} d_s \quad \text{where} \quad d_s = \sum_{j\in J_s} b_j$$

That means that, $x\gamma_2^* y$ iff $x\beta_\cdot^* y$ ∎

3.1.4 REMARK

Using Propositions 3.1.2 and 3.1.3 one can prove that

$(R/\gamma_1^*)/\gamma_2^* = (R/\gamma_1^*)/\beta_\cdot^*$ is an additive H_v-ring, and

$(R/\gamma_2^*)/\gamma_1^* = (R/\gamma_2^*)/\beta_+^*$ is a multiplicative H_v-ring.

This remark gives rise to the following question:

What happens if we take a number of quotients of a H_v-ring by the γ_1^* or γ_2^* relations?

We can state the following:

3.1.5 CONJECTURE

Let $\langle R,+,\cdot\rangle$ be a H_v-ring, then

$$\left[\left((R/\gamma_{i_1}^*)/\gamma_{i_2}^*\right)\cdots\right]/\gamma_{i_n}^* \cong R/\gamma^*$$

where $i_1,\cdots,i_n \in \{1,2\}$.

In the following we give some results on H_v-structures which obviously can be transferred on the analogous known hyperstructures.

Let $(R,+,\cdot)$ be a H_v-ring and let β^*, β_+^* be the fundamental relations with respect to multiplication and addition, respectively.

The quotient hyperstructures by β^*, β_+^* are H_v-rings. Denote ω_+ , ω^* the kernels of the canonical maps for the quotients by β_+^* , γ^* respectively. Then some results can be transferred for these relations, for example Theorem 1.2.9 could be stated as follows:

Let $(R,+,\cdot)$ be a H_v-ring, then

$u\gamma^*u'$ iff there exist $A,A' \subset \gamma^*(a)$, $B,B' \subset \gamma^*(b)$ for some $a,b \in R$ such that $u+A \cap B \neq \emptyset$ and $u'+A' \cap B \neq \emptyset$

Notice that the hyperproduct in R is not used in this result.

3.1.6 REMARK

The fundamental properties of the relation γ^* with respect to addition and multiplication of the equivalence classes give immediately the following property

If $u = \sum\limits_{j\in J} \left(\prod\limits_{i\in I_j} x_i \right) \in \mathfrak{U}$ then

$$\gamma^*(u) = \circledast\sum_{j\in J} \left(\circledcirc\prod_{i\in I_j} \gamma^*(x_i) \right) = \gamma^*(z) \quad \text{for all } z \in u,$$

where we denote $\circledast\Sigma$ and $\circledcirc\Pi$ the sum and the product of classes.

3.1.7 DEFINITIONS

Let R be a H_v-ring, then denote by β, β_+ the relations defined in R as follows

$x\beta y$ iff $\exists (z_1,\ldots,z_n) \in R^n$ such that $\{x,y\} \subset z_1 \cdots z_n$

and

$x\beta_+ y$ iff $\exists (z_1,\ldots,z_n) \in R^n$ such that $\{x,y\} \subset z_1+\ldots+z_n$

where the $z_1 \cdots z_n$ and $z_1+\ldots+z_n$ are referred to some patterns of parentheses.

Denote by β^* , β_+^* the transitive closure of the relations β, β_+, respectively. We call β^*, β_+^* **fundamental relations** with respect to multiplication and addition, respectively. Denoting by $\beta^*(a)$, $\beta_+^*(a)$ the equivalence classes of the element $a \in R$, we have

$\beta^*(a) \subset \gamma^*(a)$, $\beta_+^*(a) \subset \gamma^*(a)$, for every a of R.

In the following we will denote by

$$\varphi : R \longrightarrow R/\beta^* : x \longrightarrow \varphi(x) = \beta^*(x)$$

$$\varphi_+ : R \longrightarrow R/\beta_+^* : x \longrightarrow \varphi_+(x) = \beta_+^*(x)$$

$$\varphi^* : R \longrightarrow R/\gamma^* : x \longrightarrow \varphi^*(x) = \gamma^*(x)$$

the canonical maps defined for all $x \in R$.
Remark that the maps $\varphi : (R,+) \longrightarrow (R/\beta_+^*, \oplus)$, $\varphi_+ : (R,\cdot) \longrightarrow (R/\beta^*, \odot)$, and $\varphi^* : (R,+,\cdot) \longrightarrow (R/\gamma^*, \oplus, \odot)$ are (strong) homomorphisms.

Let ω_+, ω^* be the kernels of φ_+ and φ^* respectively, so if we denote by $\bar{0}$ the zero element of R/β_+^* or R/γ^* then

$$\omega_+ = \ker\varphi_+ = \{ x \in R : \varphi_+(x) = \bar{0} \}$$

$$\omega^* = \ker\varphi^* = \{ x \in R : \varphi^*(x) = \bar{0} \} .$$

Notice that $\omega_+ \subset \omega^*$. From now on we will use ω^* to denote also the element $\bar{0}$ of R/γ^*. Moreover, if there exists a unit element $\bar{1}$ in the semigroup $(R/\beta^*, \odot)$ then we will denote by ω the set

$$\omega = \{ x \in R : \varphi(x) = \bar{1} \} .$$

3.1.8 THEOREM

Let $(R,+,\cdot)$ be a H_v-ring, then

(i) $R\omega^* \subset \omega^*$, $\omega^* R \subset \omega^*$

(ii) If $(R,+)$ is a regular hypergroup then ω^* is a hyperideal of R.

Proof

(i) For all $a \in rx$ with $r \in R$, $x \in \omega^*$ we have

$$\varphi^*(a) = \varphi^*(rx) = \varphi^*(r) \odot \varphi^*(x) = \varphi^*(r) \odot \bar{0} = \bar{0}.$$

(ii) First, if $a,b \in \omega^*$ i.e. $\varphi^*(a) = \varphi^*(b) = \omega^*$ then

$$\varphi^*(a+b) = \varphi^*(a) \oplus \varphi^*(b) = \omega^* \oplus \omega^* = \omega^*, \text{ so } a+b \in \omega^*.$$

Let us suppose that $(R,+)$ is regular and let e be an identity of it. Then $e \in \omega_+ \subset \omega^*$. Now let us take $x \in \omega^*$. Then for all $x' \in R$ such that $e \in x+x'$, i.e. for all opposites of x, we have

$$\omega^* = \varphi^*(e) = \varphi^*(x+x') = \varphi^*(x)\oplus\varphi^*(x') = \omega^*\oplus\varphi^*(x').$$

So $\varphi^*(x') = \omega^*$ and thus $x'\in\omega^*$.

Therefore, for every $y\in\omega^*$ we have

$$y\in e+y \subset (x+x')+y = x+(x'+y)$$

and since $x'+y\in\omega^*$ we have $y\in x+\omega^*$. So $\omega^* \subset x+\omega^*$, consequently ω^* is a subhypergroup of $(R,+)$. ■

3.1.8 THEOREM

For all additive H_v-rings we have $\gamma^* = \beta^*_+$.

Proof

In an additive H_v-ring R every product of elements of R is a singleton. Thus, for every

$$u = \sum_{j\in J} \left(\prod_{i\in I_j} x_i \right) \in \mathcal{U}$$

we can consider the elements $y_j = \prod_{i\in I_j} x_i$, $j\in J$, of R, for which we have $u = \sum_{j\in J} y_j$. That means that

$$a\gamma^*b \quad \text{iff} \quad a\beta^*_+b. \quad ■$$

Since the hyperfield in the sense of Krasner is an additive hyperring, the above theorem can be applied to those hyperfields.

We know that the β^* is defined in any hypergroupoid $(R,\cdot)$. The main property is the fundamental one. For the same reason, the γ^* can be defined in hyperstructures weaker than H_v-semiring. For example, the WASS law does not need to be valid for $(+)$ or for $(\cdot)$, the reproduction axiom does not need to be valid for $+$, etc. The most important thing is that the fundamental property is still true. Using this remark we can prove the following:

3.1.10 THEOREM

Let $(R,+,\cdot)$ be a H_v-ring, then

$$R/\gamma^* \cong (R/\beta^*)/\beta^*_\oplus$$

where $\beta^*_\oplus$ is the fundamental relation defined in $(R/\beta^*,\oplus)$ by setting

$$\beta^*(a)\oplus\beta^*(b) = \{\beta^*(c):\ c\in\beta^*(a)+\beta^*(b)\}$$

Proof
The additive H_v-structure $(R/\beta^*,\oplus,o)$ divided by $\beta^*_\oplus$, Theorem 3.1.9, gives a ring. Let us denote by σ the equivalence relation associated to the projection

$$f: R \longrightarrow (R/\beta^*)/\beta^*_\oplus$$

Then, since f is a ring homomorphism, we obtain that $\sigma(a)\supset\gamma^*(a)$ for all $a\in R$.
On the other hand, since $\beta^*(x)\subset\gamma^*(x)$ for all $x\in R$, we have

$$\bigsqcup_{\beta^*(z)\in\beta^*(x)\oplus\beta^*(y)} \beta^*(z) = \bigsqcup_{z\in\beta^*(x)+\beta^*(y)} \beta^*(z) \subset \bigsqcup_{z\in\gamma^*(x)+\gamma^*(y)} \gamma^*(z)$$

Yet, from the fundamental property in $(R/\gamma^*,\boxplus,\boxdot)$, we know that $\gamma^*(x)\boxplus\gamma^*(y)$ is a singleton, so $\gamma^*(x)+\gamma^*(y)=\gamma^*(w)$ where $w\in x+y$. Therefore,

$$\bigsqcup_{\beta^*(z)\in\beta^*(x)\oplus\beta^*(y)} \beta^*(z) \subset \gamma^*(w), \quad w\in x+y.$$

Consequently, for every finite sum of elements in R/β^*, we have

$$\bigsqcup_{z\in\oplus\sum_{i\in I}\beta^*(x_i)} \beta^*(z) \subset \gamma^*(w), \quad \text{where } w \in +\sum_{i\in I} x_i$$

Moreover, since γ^* is transitively closed, we have

$$\sigma(a) = \bigsqcup_{\{z:(\beta^*(z))\beta^*_\oplus(\beta^*(a))\}} \beta^*(z) \subset \gamma^*(a) \quad \text{for all } a\in R.$$

Therefore, $\sigma = \gamma^*$. ∎

3.2 H_v-RINGS, H_b-RINGS

H_b-structures can be obtained from a corresponding structure by setting, in some of the results, greater subsets. Therefore, in order to obtain H_b-rings one can replace at least one of the sums or one of the products of a given ring, with any set containing the corresponding sum or product. If only one sum or product is replaced by a greater subset, then it yields a very thin H_v-ring.

3.2.1 THEOREM

Let $(S_i,+,\cdot)$, $i\in I$, $|I|>1$ be a pairwise disjoint family of H_v-rings then in the set

$$S = \bigcup_{i\in I} S_i$$

two hyperoperations $\oplus,\odot$, are defined, called **S-hyperoperations**, as follows:

$$x_i\oplus y_i = x_i+y_i \ , \quad x_i\odot y_i = x_iy_i \quad \text{for all} \quad (x_i,y_i)\in S_i^2$$

$$x_i\oplus x_j = x_i\odot x_j = S_i\cup S_j \quad \text{for all} \quad (x_i,x_j)\in S_i\times S_j, \quad i\neq j.$$

Then $(S,\oplus,\odot)$ is a H_v-ring which will be called **H_v-ring obtained by S-construction.**

Proof

From the S-construction 1.5.10 it is clear that $(S,\oplus)$ is a H_v-group.

Moreover, $\odot$ is WASS.

Now, since every $(S_i,\oplus,\odot)$ is H_v-ring, for all x_i,y_i,z_i in S_i it is obtained

$$(x_i\odot(y_i\oplus z_i))\cap((x_i\odot y_i)\oplus(x_i\odot z_i)) \neq \emptyset$$

Finally, for all $(x_i,x_j,x_k) \in S_i\times S_j\times S_k$ where the $i=j=k$ is not valid, then

$$x_i\odot(x_j\oplus x_k) = S_i\cup S_j\cup S_k = (x_i\odot x_j)\oplus(x_i\odot x_k)$$

Therefore, $(S,\oplus,\odot)$ is H_v-ring. ∎

Remarks

Obviously, the S/γ^* is a singleton. The S-construction can be defined even if $(S_i,+)$ are H_v-semigroups. Finally, if $(S_i,+,\cdot)$ is H_v-ring such that $(S_i,\cdot,+)$ is also H_v-ring for every $i\in I$, then the dual hyperstructure $(S,\odot,\oplus)$ of $(S,\oplus,\odot)$, is also a H_v-ring.

3.2.2 THEOREM (Associated H_v-rings)

Let $(H,\cdot)$ be a H_v-group, then for every hyperoperation $\oplus$ such that

$$x\oplus y \supset \{x,y\} \quad \text{for all} \quad x,y\in H,$$

the hyperstructure $(H,\oplus,\cdot)$ is a H_v-ring.

Proof
For all $x,y,z\in H$ we have

$$[(x\circ y)\circ z \cap x\circ(y\circ z)] \supset \{x,y,z\} \neq \emptyset$$

$$x\circ H = \bigcup_{h\in H} x\circ h \supset \bigcup_{h\in H} \{x,h\} = H.$$

Moreover,

$$x(y\circ z) \supset x\{y,z\} = xy\cup xz \quad \text{and} \quad xy\circ xz \supset (xy\cup xz).$$

So, $x(y\circ z) \cap (xy\circ xz) \neq \emptyset$. Thus $(H,\circ,\cdot)$ is a H_v-ring. ∎

All the above H_v-rings will be called **associated to $(H,\cdot)$ H_v-rings.**

3.2.3 REMARK

The associated H_v-rings can be used for the one dimensional representations. For the one dimensional representations of hypergroups one can use the following types of associated hyperrings:

Let $(H,\cdot)$ be a hypergroup, then the following three types of associated hyperrings $(H,\circ,\cdot)$ can be defined, for all $x,y\in H$

type a : $x\circ y = \{x,y\}$

type b : $x\circ y = \beta^*(x)\cup\beta^*(y)$

type c : $x\circ y = H$

Proof
The reproductivity for all types is obvious, for the rest axioms we have for all $x,y,z\in H$, the following

Type a: $x\circ(y\circ z) = \{x,y,z\} = (x\circ y)\circ z$ and

$$x(y\circ z) = xy\cup xz = (xy)\circ(xz)$$

so the strong distributivity is valid.

Type b: $x\circ(y\circ z) = \beta^*(x)\cup\beta^*(y)\cup\beta^*(z) = (x\circ y)\circ z$ and

$$x(y\circ z) = x(\beta^*(y)\cup\beta^*(z)) \subset \beta^*(x)(\beta^*(y)\cup\beta^*(z)) = \bigcup_{\substack{u\in xy\\ v\in xz}} (\beta^*(u)\cup\beta^*(v))$$

$$(xy)\circ(xz) = \bigcup_{u\in xy,\, v\in xz} (\beta^*(u)\cup\beta^*(v))$$

so the inclusion distributivity is valid.

Type c: $x\oplus(y\oplus z) = H = (x\oplus y)\oplus z$ and

$x(y\oplus z) = H = (xy)\oplus(xz)$

so the strong distributivity is valid. ■

Moreover, the complete above cases can be defined as follows: For all types a,b,c, consider the set $H' = H\cup\{0\}$, where $0\notin H$, instead of H, and extend the above hyperoperations on H' by setting

$$0\oplus x = x\oplus 0 = x, \quad 0x = x0 = 0 , \quad \forall x\in H'.$$

One can see that $(H',\oplus,\cdot)$ is always a hypersemiring where for the complete types a and c the strong distributive law is valid.

3.2.4 THEOREM

Let $\langle R_o,+,\cdot\rangle$ be a H_v-semiring and $0_1,\ldots,0_n$ be elements not in R_o. Consider the sets

$$R_{i+1} = R_i \cup \{0_{i+1}\} \quad \text{for} \quad i = 0,\ldots,n-1$$

Then $\langle R_n,+,\cdot\rangle$ becomes a H_v-semiring where the hyperoperations $(+),(\cdot)$ are defined by extending them on the new elements with the relations

$$\left.\begin{aligned} 0_i + r &= r + 0_i = r \\ 0_i \cdot r &= r \cdot 0_i = 0_i \end{aligned}\right\} \quad \text{for all} \quad i\in\{1,\ldots,n\} , \ r\in R_o \quad \text{and}$$

$$\left.\begin{aligned} 0_i + 0_j &= 0_j + 0_i = 0_j \\ 0_i \cdot 0_j &= 0_j \cdot 0_i = 0_i \end{aligned}\right\} \quad \text{if} \quad i \geq j .$$

Proof. Immediate. ■

The above H_v-semirings can be used in the representation theory of hypergroups by hypermatrices. In the series of H_v-semirings $R_o,R_1,\ldots,R_n$ each R_{i+1} can be obtained from R_i by attaching a new zero element 0_{i+1}.

In the special case when $R_o = \{0\}$ the above semi(hyper)ring

is isomorphic to the Boolean semiring defined on the ascending series of sets $X_o \subset X_1 \subset \ldots \subset X_n$ where $(+)$ and $(\cdot)$ are the intersection and the union respectively.

One can see that the γ^* classes of the elements of R_o remain the same in R_n and all the elements $0_1,\ldots,0_n$ belong to the S_{R_o} i.e. they are singles.

3.3 ABSORBING ELEMENTS

3.3.1 DEFINITION

Let $(R,+,\cdot)$ be a H_v-ring. An element $a\in R$ will be called **left absorbing** element if there exists an element d of R, which will be called **left destination** element, such that $aR=d$. Similarly, if $Ra=d$ then a will be called **right absorbing** element and d will be called **right destination** element. If $aR=d_1$ and $Ra=d_2$ then a will be simply called **absorbing** element. If $S\subset R$, $|S|>1$ and $aS=d$, then a will be called element **absorbing the set S from the left.** Similarly, for the right and the both sides.

This definition can be given for any H_v-semigroup $(R,\cdot)$.

Let A_ℓ , A_r denote the sets of left and right absorbing elements and D_ℓ , D_r the sets of left and right destination elements respectively.

3.3.2 PROPOSITION

The left destination elements are left absorbing elements, i.e. $D_\ell \subset A_\ell$. Similarly $D_r \subset A_r$.

Proof

Let $aR=d$, then $dR = aRR \subset aR = d$. Therefore d is a left absorbing element with itself as destination. ∎

3.3.3 THEOREM

Let $(R,+,\cdot)$ be H_v-ring. If $D_\ell \cap D_r \neq \emptyset$ then $D_\ell \cup D_r = \{d\}$ is a singleton, it means that in this case there is a unique destination.

Proof

Let $d_1\in D_\ell$, $d_2\in D_r$ then there are $a_1\in A_\ell$, $a_2\in A_r$ for which we have

$$d_1 = a_1R = a_1a_2 = Ra_2 = d_2 .$$

That proves that all left and right destination elements are equal. So we can write $D_\ell \cup D_r = d$ and we can say that there exists the destination d. ∎

3.3.4 REMARK

Let $d \in D_\ell$ then there exists $a \in A_\ell$ such that $aR=d$. Take $x,y \in R$, then $a(x+y)=d$ and $ax+ay=d+d$. But $a(x+y) \cap (ax+ay) \neq \emptyset$ so $d \in d+d$. This means that only the elements such that $d \in d+d$ can be destination elements. Moreover, since d is a partial identity, it is an element of the core ω_R; thus $D_\ell \cup D_r \subset \omega_R$. This result can be obtained also from the fact $d^2=d$.

If the strong distributivity is valid then $d=d+d$.

3.3.5 PROPOSITION

Let I_ℓ be a left H_v-ideal of the H_v-ring $(R,+,\cdot)$, then $D_r \subset I_\ell$. Similarly, if I_r be a right H_v-ideal, then $D_\ell \subset I_r$.

Proof
Let $d \in D_r$ then $d \in A_r$ so $Rd=d$ and $I_\ell d=d$. But since I_ℓ is left H_v-ideal, $I_\ell R \subset I_\ell$ which imlies $I_\ell d \subset I_\ell$. Therefore, $d \in I_\ell$. ∎

3.4 H_v-MODULES

Recall from Definition 1.2.8 that V is a left H_v-vector space (respectivelly H_v-module) over a H_v-field (resp. H_v-ring) F if $(V,+)$ is a COW H_v-group, and there exists $\cdot : F \times V \longrightarrow \mathcal{P}(V)$: $(a,x) \longmapsto ax$ such that, for all $a,b \in F$ and $x,y \in V$, we have:

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset \qquad (I)$$

Moreover, the fundamental relation ε^* is defined to be the smallest equivalence relation such that the quotient V/ε^* is a vector space (resp. module) over the field (resp. ring) F/γ^*.

Remark that if V contains an absolute zero 0 for which $a0=0$, then the above weak properties (I) are always valid for this element.

3.4.1 REMARK

One can prove analogous to Theorems 1.2.2 and 1.2.7 for the ε^*. So ε^* is the transitive closure of the relation ε defined in V by setting

$$x\varepsilon y \quad \text{iff} \quad \{x,y\}\subset u$$

where $u\in\mathcal{U}$ and $\mathcal{U}$ is the set of all expressions consisting of finite hyperoperations of either on F and V or of the external hyperoperation, applying on finite sets of elements of F and V. Then let $(V,+)$ be a H_v-vector space over F. Denote $\sum a_i v_i$ where $a_i v_i$ is either the product $a_i v_i$, if $a_i\in F$, or the element v_i, if a_i does not appear. Then, from the fundamental property, it is immediate that

$$\varepsilon^*(\textstyle\sum a_i v_i) = \sum \gamma^*(a_i)\varepsilon^*(v_i).$$

Denoting β^* the fundamental class in $(V,+)$ then

$$\beta^*(v) \subset \varepsilon^*(v) \quad \text{for all } v \text{ in } V.$$

The ε^* class of an element is greater than the β^* class because two elements $v_1, v_2\in V$, which are not β^* equivalent, can be ε^* equivalent if they belong to a product ax with $a\in F$, $v\in V$.

Now, it is easy to see that the following theorem is valid.

3.4.2 THEOREM

If for all $a\in F$ and $v\in V$ there exists a $u\in V$ such that

$$\gamma^*(a)\beta^*(v) \subset \beta^*(u) \quad \text{then} \quad \varepsilon^* = \beta^*.$$

3.4.3 DEFINITION

For $\mathcal{U}\subset V$, let $[\mathcal{U}]$ denote the set of all elements which belong to the sets of the form $\sum a_i v_i$, $v_i\in\mathcal{U}$ in the notation of the Remark 3.4.1. If $[\mathcal{U}]=V$, then $\mathcal{U}$ **generates** V. If there is no element $u\in\mathcal{U}$ such that $u\in[\mathcal{U}-\{u\}]$, then $\mathcal{U}$ is called **(linearly) independent** subset of V.

3.4.4 THEOREM

If the vectors $\varepsilon^*(v_1),\ldots,\varepsilon^*(v_s) \in V/\varepsilon^*$ are independent over F/γ^* then all $v'_1,\ldots,v'_s \in V$ where $v'_t \in \varepsilon^*(v_t)$, $t=1,\ldots,s$, are independent over F.

Proof

Suppose that there exists $v'_t \in [V_t]$ where $V_t=\{v'_1,\ldots,v'_s\}-\{v'_t\}$.

Therefore, there exists a_i's such that $v'_t \in \sum a_i v'_i$ where $v'_i \in V_t$.

So from the Remark 3.4.1 it is obtained

$$\varepsilon^*(\sum a_i v'_i) = \sum \gamma^*(a_i)\varepsilon^*(v'_i)$$

which is a contradiction. ∎

3.4.5 REMARKS

In a H_v-vector space one can define two kinds of single elements: one with respect to ε^* and one with respect to β^* in $(V,+)$, obviously $\beta^*(v) \subset \varepsilon^*(v)$ for all v in V. However, from the fundamental property and the Remark 1.3.4, if $\varepsilon^* \neq \beta^*$, then there are no single elements with respect to both ε^* and β^*. This is so, because in this case, if s is a common single element then

$$v+s = \varepsilon^*(v+s) = \beta^*(v+s).$$

Thus the ε^* and β^* classes are determined by the sets $v+s$, $v \in V$.
In this case, as in Theorem 3.4.2, all the products of the form λv where $\lambda \in F, v \in V$, contain elements of V which belong to the same fundamental β^*-class.

These remarks are necessary for constructions of H_v-vector spases with single elements. In particular for constructions where $\varepsilon^*(u)=0$, i.e. the zero class is a single element of V. So in $(V,+)$ there exists a zero element 0 and this is unique. In this case any "linear expression" $\sum a_i v_i$ containing the 0 satisfies the condition $\sum a_i v_i = 0 \in V$. Therefore, 0 is an absorbing element in F and in the external hyperproducts. More precisely $a0=0$ for all $a \in F$, indeed it is obtained

$$\gamma^*(a)\cdot\varepsilon^*(0) = \varepsilon^*(0), \quad \text{thus} \quad \gamma^*(a)\cdot 0 = 0.$$

3.5 A MATRIX P-HYPERALGEBRA

In this paragraph we define a hyperalgebra on a set of non-square matrices, which is also an algebra, using the P-hyperoperations.

3.5.1 DEFINITION

Let $M = M_{m\times n}$ be the module of $m\times n$ non-square matrices over the ring R and let $P = \{ P_i : i\in I \} \subseteq M$. We define a hyperoperation P^* on M as follows

$$P^* : M\times M \longrightarrow \mathcal{P}(M) : (A,B) \longmapsto AP^*B = \{ AP_i^tB : i\in I \} \subset M$$

The hyperoperation P^*, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. It is easy to see that P^* is strong associative and the inclusion distributive law with respect to addition of matrices

$$AP^*(B+C) \subseteq AP^*B+AP^*C \quad \text{for all } A,B,C \text{ in } M$$

is valid. Therefore, $(M,+,P^*)$ defines a multiplicative hyperring on non-square matrices. If R is a field then M becomes a hyperalgebra which is called [35], **P-hyperalgebra** of the set of $m\times n$ matrices over R. Of course, more P-hyperoperations can be defined: for the sum of the elements of M and for the external operation from R to M.

Finally, remark that one can face the above problem as follows: Find a multiplicative P-hyperring R and hypermatrices over R such that the above hyperproduct could be obtained.

3.5.2 REMARK

One generalization of the above P^*-hyperoperation is the following:

Let $$M = \bigcup_{k_1\le m,n\le k_2} M_{m\times n}$$

be the set of all $m\times n$ matrices over R indexed from $I\times I$ where $I=\{k_1,\ldots,k_2\}$. Let us consider any set $P\subset M$ such that $P\cap M_{m\times n}\neq\emptyset$ for every $(m,n)\in I\times I$. Then we define the hyperoperation P^*, for every $A\in M_{m\times n}$ and $A\in M_{r\times s}$ as follows

$$AP^*B = \{ AP_vB : P_v\in P\cap M_{n\times r} \}$$

It is immediate that this hyperoperation is well defined on any complete set $I\times I$ of matrices, of any size up to I. Moreover, P^* is associative, thus M becomes a semihypergroup.

If $|P\cap M_{m\times n}|=1$ for all $(m,n)\in I\times I$ then P^* is actually an operation.

However, the set M is not an algebra since the addition is still a partial operation.

3.5.3 THEOREM

Denoting by γ^* the fundamental relation with respect to addition and the P-hyperoperation, i.e. the fundamental relation on the hyperring $(M,+,P^*)$, we have $\varepsilon^*=\gamma^*$.

Proof
It is enough to prove that $\varepsilon^*(x)\subset\gamma^*(x)$ for all $x\in M$. Recall the definition of the set $\mathcal{U}$ in Remark 3.4.1. From the fact that both ε^* and γ^* are transitively closed, it is enough to prove that every $u\in\mathcal{U}$ can be written as finite polynomial of elements of M. The external product in M over R is an operation. Therefore, if a coefficient $\lambda\in R$ appears in one $u\in\mathcal{U}$ multiplied by a matrix A, then one can replace A by λA and a polynomial in M can be obtained.

With this the proof of the theorem is complete. ■

Πάντα η πραγματικότητα
μου γνέφει πρωτόγνωρα
και ζω.

Reality always
nods to me
like a new acquaintance
and I live.

CHAPTER 4

UNITING ELEMENTS

4.1 UNITING ELEMENTS. ASSOCIATIVITY, COMMUTATIVITY, REPRODUCTIVITY

In [6] a new method was introduced to obtain stricter domains from given ones.

4.1.1 DEFINITION

The following method is called **uniting elements:**

Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is a H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)/\beta^*$ for which the property d is valid, is obtained.

(I) The associativity

Let $(G,\cdot)$ be a groupoid. Uniting elements of the transitive closure of the "product elements" one obtains that the quotient set is a H_v-semigroup which the quotient by β^* is a semigroup. (For complete proofs in this case see Corsini's book [5]).

(II) The commutativity

Let $(G,\cdot)$ be a groupoid. We unite two elements a and b of G if there is a pair $(x,y)\in G^2$ such that $xy=a$, $yx=b$, and we take the transitive closure. Then it is clear that the quotient set is a H_v-commutative groupoid so, divided by β^*, a commutative groupoid is obtained.

(III) The reproductivity

4.1.2 THEOREM

Let $(S,\cdot)$ be a commutative semigroup which has at least one element u such that the set uS is finite. Consider the transitive closure $\mathcal{R}^*$ of the relation $\mathcal{R}$ defined as follows:

$$s_1 \mathcal{R} s_2 \quad \text{iff} \quad \exists x\in S \text{ such that } xs_1 = xs_2 .$$

Then $\langle S/\mathcal{R}^*, \circ\rangle/\beta^*$ is a finite commutative group.

Proof

Let's suppose that a and b are equivalent elements of S, i.e. $a\mathcal{R}^*b$. That means that there are elements $x_1,\dots,x_{n+1}$, $\mu_1,\dots,\mu_n$ of S such that

$$x_1 a = x_1\mu_1,\ x_2\mu_1 = x_2\mu_2,\ \dots,\ x_n\mu_{n-1} = x_n\mu_n,\ x_{n+1}\mu_n = x_{n+1}b .$$

From these relations it is obtained that

$$x_{n+1}\dots x_2x_1a = x_{n+1}\dots x_2x_1\mu_1 = x_{n+1}\dots x_2x_1\mu_2 = x_{n+1}\dots x_2x_1b$$

so, setting $x=x_{n+1}\dots x_2x_1$, we have $xa=xb$.

Therefore it is proved that $\mathcal{R}^*\subseteq\mathcal{R}$.

In the following let's denote by $\bar{a}$ the class of the element a and $\bar{S} = S/\mathcal{R} = S/\mathcal{R}^*$. Consider the mapping

$$\mathcal{I} : \bar{S} \longrightarrow \{\ \bar{z} \mid z\in xS\ \} : \bar{s} \longmapsto \overline{xs}$$

$\mathcal{I}$ is a bijection because

$$\bar{s}=\bar{s}' \longrightarrow s\mathcal{R}s' \longrightarrow \exists p:\ ps=ps' \longrightarrow xps=xps' \longrightarrow$$

$$\longrightarrow p(xs)=p(xs') \longrightarrow \overline{xs}=\overline{xs}'$$

Vice versa let $\overline{ys}=\overline{ys}'$. Then there exists q such that

$$qys = qys' \longrightarrow (qy)s = (qy)s' \longrightarrow \bar{s}=\bar{s}'$$

so $\mathcal{I}$ is a bijection.

From the above we have

$$\aleph_o > |uS| \geq |\{\ \bar{z}:\ z\epsilon xS\ \}| = |\bar{S}|.$$

Consider the subset $\overline{xS} = \{\overline{xs}: x\epsilon S\}$ of $\bar{S}$. Then we have $|\overline{xS}|=|\bar{S}|$ and since $|\bar{S}|<\aleph_o$ it follows that $\overline{xS} = \bar{S}$. So for every $\bar{y}\epsilon\bar{S}$, there exists a $z\epsilon S$ such that $\bar{y}=\overline{xz}$, but $xz\epsilon\bar{x}\cdot\bar{z}$, so $\overline{xz}\epsilon\bar{x}\circ\bar{z}$ or $\bar{y}\epsilon\bar{x}\circ\bar{z}$.

Therefore the reproduction axiom is satisfied in $\langle\bar{S},\circ\rangle$ and S/β^* is a finite commutative group. ■

For the above equivalence we have the following.

4.1.3 THEOREM

Let $x,a\epsilon S$, then the following function is a bijection

$$f : \bar{S} \longrightarrow \bar{x}\circ\bar{S} : \bar{a} \longmapsto \bar{x}\circ\bar{a}$$

Proof
We have, $\bar{a}=\bar{b}$ implies $\bar{x}\cdot\bar{a}=\bar{x}\cdot\bar{b}$ implies $\bar{x}\circ\bar{a}=\bar{x}\circ\bar{b}$.

Vice versa, let $\bar{x}\circ\bar{a} = \bar{x}\circ\bar{b}$ that is

$$\forall u\epsilon\ \bar{x}\cdot\bar{a},\ \exists\ w\epsilon\ \bar{x}\cdot\bar{b}:\ \bar{u}=\bar{w}\ \text{ and} \qquad (1)$$

$$\forall w'\epsilon\bar{x}\cdot\bar{b},\ \exists\ u'\epsilon\bar{x}\cdot\bar{a}:\ \bar{u}'=\bar{w}' \qquad (2)$$

Using only the (1) i.e. $\bar{x}\circ\bar{a} \subset \bar{x}\circ\bar{b}$, we shal prove that $\bar{a}=\bar{b}$.

Let $u=\lambda\mu$, $\lambda\epsilon\bar{x}$, $\mu\epsilon\bar{a}$ so $\exists(p,q)$: $p\lambda=px$, $q\mu=qa$. Then from hypothesis

$$\exists\ \bar{w}\ \epsilon\ \bar{x}\circ\bar{b}:\ \bar{w} = \bar{u}.$$

Let's suppose that $w \in \bar{x}\cdot\bar{b}$ then

$$w = \lambda'\mu',\ \lambda'\epsilon\bar{x},\ \mu'\epsilon\bar{b}$$

so $\exists(t,s)$: $t\lambda'=tx$, $s\mu'=sb$ $\exists f$: $fu=fw$.

Moreover $\{\lambda,\lambda'\} \subset \bar{x} \longrightarrow \exists g:\ g\lambda = g\lambda'$.

From $fu=fw$, $u=\lambda\mu$, $w=\lambda'\mu'$ one obtains $fq\lambda\mu = fq\lambda'\mu'$ and since $q\mu=qa$ we have $f\lambda qa = fq\lambda'\mu'$.

Using the equality $s\mu'=sb$ one obtains

$$sf\lambda qa = sf\lambda' q\mu' = f\lambda' q(s\mu') = f\lambda' qsb.$$

Then since $g\lambda=g\lambda'$ we obtain from $fqsg\lambda a = fqs\lambda' b$ that

$(fqsg\lambda)a = (fqsg\lambda)b$ which means that $a=b$.

Therefore the mapping f is a bijection. ∎

4.1.4 THEOREM

If $S=\langle a\rangle$ is a finite cyclic semigroup of index r and period m, then if $\mathcal{R}$ is the above equivalence relation we have

$$\langle S/\mathcal{R}, \circ\rangle \cong \mathbb{Z}_m$$

Proof

We have $S = \{ a, a^2, \ldots, a^r, \ldots, a^{r+m-1}\}$ and $a^{r+m} = a^r$.

Let $h\in\mathbb{N}$, $h<r+m$ and

$$k_h = \min\{ j \mid h+jm\geq r \} .$$

Then $h+k_h m = r+t$ where $0\leq t<m$. We have

$$a^{h+mak_h m} = a^h(a^{k_h m}a^m) = a^{h+k_h m}a^m = a^{r+t+m} = a^{r+t} = a^{h+k_h m} = a^h a^{k_h m}.$$

So it has been proved that $a^h \mathcal{R} a^{h+m}$ and then $a^h \mathcal{R} a^{h+im}$, $\forall i\geq 0$.

If $m=1$, then $a\mathcal{R}a$ for all $k\in\mathbb{N}_o$ and $\langle S/\mathcal{R}, \circ\rangle \cong \mathbb{Z}/\mathbb{Z} = \mathbb{Z}_1$.

Let us suppose now that $m\geq 2$ and there are h,k such that $1\leq k<h\leq m$, $a^h \mathcal{R} a^k$, i.e. $\exists q:\ a^q a^h = a^q a^k$. We have

$$a^r a^q a^h a^{m-k} = a^r a^q a^k a^{m-k} \quad \text{so} \quad a^q a^m a^r = a^q a^r = a^q a^r a^{m+h-k} = a^q a^r a^{h-k}.$$

Let $q=pm+q'$ where $q'<m$. We have

$$a^q a^r = a^{pm+q'} a^r = a^{q'} a^r = a^{pm+q'} a^r a^{h-k} = a^{q'} a^r a^{h-k}.$$

Hence we can always suppose $q<m$ and thus we can write

$$a^q a^r = a^q a^r a^{h-k}.$$

If $q+h-k<m$ one obtains $a^{q+r} = a^{q+h-k+r}$ which is a contradiction.

If $q+h-k=m+t$ where $0\le t<m$, it follows

$$a^{q+r} = a^{m+t+r} = a^{r+t}$$

from which $q=t$ and then $q+h-k=m+q$ which is a contradiction since $h-k<m$.
Then for all (s,t)

$$a\mathcal{R}a \quad \text{iff} \quad s-t \in m\mathbb{Z}.$$

Let be $m\ge r$, then

$$a\mathcal{R}a^{r+(m-r)+1},$$

$$a^2\mathcal{R}a^{r+(m-r)+2},$$

$$\dots\dots\dots\dots$$

$$a^{r-1}\mathcal{R}a^{r+m-r+r-1} = a^{r+m-1}.$$

Thus the equivalence classes are

$$\{a,a^{r+(m-r)+1}\},\ \{a^2,a^{r+(m-r)+2}\},\dots,\ \{a^{r-1},a^{r+m-1}\},$$

$$\{a^r\},\{a^{r+1}\},\dots,\{a^{r+(m-r)}\}.$$

Now let be $r>m$ i.e. $r=h+mq$, $h<m$ and $q>0$. We have

$$a\mathcal{R}a^{m+1}\mathcal{R}a^{2m+1}\mathcal{R}\ \dots\ \mathcal{R}a^{qm+1},$$

$$a^2\mathcal{R}a^{m+2}\mathcal{R}a^{2m+2}\mathcal{R}\ \dots\ \mathcal{R}a^{qm+2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a^h\mathcal{R}a^{m+h}\mathcal{R}a^{2m+h}\mathcal{R}\ \dots\ \mathcal{R}a^{qm+h},$$

$$a^{h+1}\mathcal{R}a^{m+h+1}\mathcal{R}a^{2m+h+1}\mathcal{R}\ \dots\ \mathcal{R}a^{qm+h+1} = a^{r+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a^{m-1}\mathcal{R}a^{m+m-1}\mathcal{R}\ \dots\ \mathcal{R}a^{r+m-h-1},$$

$$a^m\mathcal{R}a^{2m}\mathcal{R}\ \dots\ \mathcal{R}a^{qm+h+m-h} = a^{r+m-h},$$

moreover

$$a^{r+m-h+1}\mathcal{R}a,\ a^{r+m-h+2}\mathcal{R}a^2,\ \ldots,\ a^{r+m-h+(h-1)} = a^{r+m-1}\mathcal{R}a^{h-1}$$

Therefore the classes are, for $1 \le i \le m$

$$\{ a^i, a^{i+m}, \ldots, a^{i+qm} \} \cup \{ a^{(q+1)m.h(i)+1} \}$$

where $h(i)=1$ if $i<h$ and $h(i)=0$ if $i \ge h$.

It is clear that $\mathcal{R}$ is a congruence, it follows that $\langle S/\mathcal{R}, \circ \rangle$ is a semigroup, nay it is a cyclic group of order m. ∎

4.2 OTHER PROBLEMS

Consider the following question:

What happens if one apply the uniting elements procedure for two or more properties that are desirable of a quotient?

The answer for groupoids is the following theorem.

4.2.1 THEOREM

Let $(G, \cdot)$ be a groupoid and

$$F = \{ f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n} \}$$

be a system of equations on G consisting of two subsystems

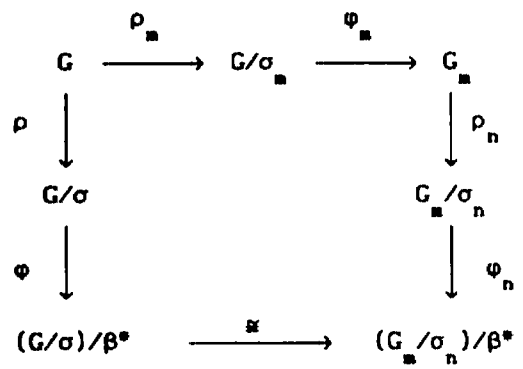
$$F_m = \{ f_1, \ldots, f_m \} \quad \text{and} \quad F_n = \{ f_{m+1}, \ldots, f_{m+n} \}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*$$

Proof.

We have the following diagram



All the mappings $\rho, \varphi, \rho_m, \varphi_m, \rho_n, \varphi_n$ are canonical so the only one we have to prove is that for all x,y in G

$$\rho\circ\varphi(x) = \rho\circ\varphi(y) \quad \text{iff} \quad \rho_m\circ\varphi_m\circ\rho_n\circ\varphi_n(x) = \rho_m\circ\varphi_m\circ\rho_n\circ\varphi_n(y)$$

i.e. an element y belongs to the associate class of x with respect to $\rho\circ\varphi$ iff y belongs to the associate class of x with respect to $\rho_m\circ\varphi_m\circ\rho_n\circ\varphi_n$.

In other words the sets

$$X = \{ z\in G:\ \rho\circ\varphi(z) = \rho\circ\varphi(x) \} \qquad \text{and}$$

$$X_n = \{ z\in G:\ \rho_m\circ\varphi_m\circ\rho_n\circ\varphi_n(z) = \rho_m\circ\varphi_m\circ\rho_n\circ\varphi_n(x) \} \qquad \text{are equal.}$$

Since $(G_m/\sigma_n)/\beta^*$ is a groupoid satisfying the system F it is clear that $X \subset X_n$.

Conversely, it is obvious that, for all x in G,

$$\{z\in G:\ \rho\circ\varphi(z)=\rho\circ\varphi(x)\} \supset \{z\in G:\ \rho_m\circ\varphi_m(z)=\rho_m\circ\varphi_m(x)\}$$

The $\rho\circ\varphi$ class satisfies the system F, thus we obtain that

$$\{z\in G:\ \rho\circ\varphi(z)=\rho\circ\varphi(x)\} \supset \{z\in G:\ \rho_m\circ\varphi_m\circ\rho_n(z)=\rho_m\circ\varphi_m\circ\rho_n(x)\}.$$

Finally from the definition of the β^* we obtain that $X \supset X_n$.

That proves the theorem. ∎

Using the same technique as in the above theorem one can prove analogous theorems for other structures. Therefore, for example, in the case of rings or semi-rings one has:

4.2.2 THEOREM

Let $(R,+,\cdot)$ be a ring (or semi-ring) and

$$F = \{ f_1,\ldots,f_m,f_{m+1},\ldots,f_{m+n} \}$$

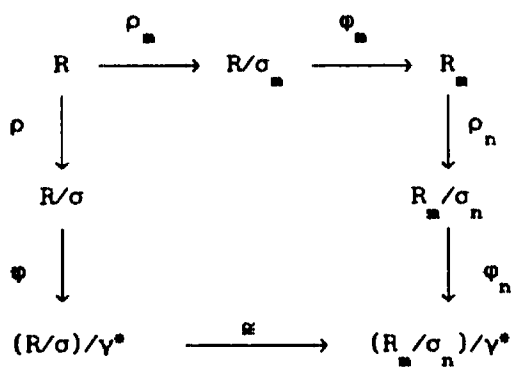
be a system of equations on R consisting of two subsystems

$$F_m = \{ f_1,\ldots,f_m \} \quad \text{and} \quad F_n = \{ f_{m+1},\ldots,f_{m+n} \}.$$

Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring (or semi-ring) $R_m = (R/\sigma_m)/\gamma^*$. Then

$$(R/\sigma)/\gamma^* \cong (R_m/\sigma_n)/\gamma^*$$

In other words we have the following commutative diagram



where all the mappings $\rho,\varphi,\rho_m,\varphi_m,\rho_n,\varphi_n$ are canonical ones.

4.3 EXAMPLES

One can unite elements, with no special intension, and can obtain very strict structures:

4.3.1 EXAMPLE

Consider the abelian semigroup $(\mathbb{N}_o,+)$, where $\mathbb{N}_o \equiv \{1,2,3,\ldots\}$. Unite the elements 1 and $n\in\mathbb{N}_o$, then it is obtained the abelian H_v-semigroup $(\bar{\mathbb{N}}_o,\oplus)$ where $\bar{\mathbb{N}}_o \equiv \{\{1,n\},\{2\},\ldots,\{n-1\},\{n+1\},\ldots\}$ and the hyperoperation $\oplus$ is given by the following multiplication table

$\oplus$	$\{1,n\}$	2	. . .	n-2	n-1	n+1	. . .
$\{1,n\}$	2,n+1,2n	3,n+2	. . .	n-1,2n-2	$\{1,n\}$,2n-1	n+2,2n+1	. . .
2	3,n+2	4	. . .	$\{1,n\}$	n+1	n+3	. . .
. . .			. . .				
n-2	n-1,2n-2	$\{1,n\}$	. . .	2n-4	2n-3	2n-1	. . .
n-1	$\{1,n\}$,2n-1	n+1	. . .	2n-3	2n-2	2n	. . .
n+1	n+2,2n+1	n+3	. . .	2n-1	2n	2n+2	. . .
. . .							. . .

Finally if we quotient by β^* it is obtained

$$\bar{\mathbb{N}}_o/\beta^* \cong \mathbb{Z}_{n-1} .$$

4.3.2 EXAMPLE

Let's consider the octadic non-commutative group of Klein: $\{1,-1,i,-i,j,-j,k,-k,\}$ defined by the relations $i^2=j^2=k^2=-1, ij=k$, $jk=i, ki=j$. In order to delete the non-commutativity we can unite the elements according the procedure sugested in 4.1(II). Then the partition

$\{ I=\{1\}, A=\{-1\}, B=\{i,-i\}, C=\{j,-j\}, D=\{k,-k\} \}$

is obtained. The quotient is a H_v-commutative groupoid where the hyperproduct $\odot$ is given by the table

$\odot$	I	A	B	C	D
I	I	A	B	C	D
A	A	I	B	C	D
B	B	B	I,A	D	C
C	C	C	D	I,A	B
D	D	D	C	B	I,A

Then the β^*-classes are $a=\{I,A\}$, $b=\{B\}$, $c=\{C\}$, $d=\{D\}$, where the identity element is a and

$b^2=c^2=d^2=a$, $bc=d$, $cd=b$, $bd=c$ (Klein's group).

4.3.3 EXAMPLE

Let $(G,\cdot)$ be a group and $S\subset G$ be a set of generators i.e. $\langle S\rangle=G$. Take the partition $\mathfrak{R}$ where the only one class with cardinality >1 is the set S. Then $(G/\mathfrak{R})/\beta^*$ is a cyclic group.

Οι δολοφόνοι των ιδανικών μας
ηγέτες της προδοσίας
έχουν το θράσος να μιλούν
για φυσικό θάνατο των ιδανικών.

The murderers of our ideals
leaders of treason
dare to talk of
the "natural death" of ideals.

CHAPTER 5

H_v - HYPERGROUPOID ALGEBRA

5.1 FUNDAMENTAL SETS

5.1.1 DEFINITIONS

Let $(G,\cdot)$ be a hypergroupoid. We call **set of fundamental maps on G with respect to $(\cdot)$**, the set

$$\theta = \{\ \theta : G\times G \xrightarrow{\text{onto}} G : (x,y) \longmapsto z \,|\, z \in xy\}$$

For H_v-groups we have $\theta \neq \emptyset$. Any subset $\theta_\mu \subset \theta$ defines a hyperoperation $(\circ_\mu)$ on G as follows

$$x \circ_\mu y = \{\ z : z = \theta(x,y) \ \text{ for some } \ \theta \in \theta_\mu\ \}$$

Obviously, $\circ_\mu \leq \cdot$ and $\theta_\mu \subset \theta_{\circ_\mu}$, where $\theta_{\circ_\mu}$ denotes the fundamental

maps on G with respect to $\circ_\mu$. A set $\theta_\alpha \subset \theta$ is called associative (respectively WASS) if and only if for every subset $\theta_\mu \in \theta_\alpha$ the hyperoperation $\circ_\mu$ is associative (resp. WASS). A hypergroupoid $(G,\cdot)$ will be called θ-WASS if there exists an element $\theta_\circ \in \theta$ which defines an associative operation $(\circ)$ in G. That means that $(G,\cdot)$ is a H_v-semigroup.

5.1.2 REMARK

For a hypergroupoid $(G,\cdot)$ with $|G|=n$ we, obviously, have

$$\#\theta \leq \prod_{(x,y)\in G^2} (\#xy)$$

Moreover, for every $\theta\in\theta$ we have

$$1 \leq |\theta^{-1}(g)| \leq n^2-n+1 \quad \text{for all} \quad g\in G$$

where $\theta^{-1}(g)$ is the inverse image of g.

However, for every $\theta\in\theta$ we have $\sum_{g\in G} |\theta^{-1}(g)| = n^2$.

5.1.3 REMARKS

(a) The property $G\cdot G=G$ in a hypergroupoid $(G,\cdot)$ does not guarantee that $\theta \neq \emptyset$. On the other side we remarked above that if G is H_v-group then $\theta \neq \emptyset$. Indeed, a construction of an element $\theta\in\theta$ is the following: For every $g\in G$, using the reproduction axiom, take a $g'\in G$ such that $g\in gg'$, so $\theta(g,g')=g$. For the rest $(x,y)\in G^2$ set $\theta(x,y)=z$ where, simply, $z\in xy$.

(b) Obviously if $(G,\cdot)$ is θ-WASS then every greater hypergroupoid is θ-WASS.

(c) All H_b-groups are θ-WASS.

(d) Any H_b-semigroup in which the b-structure has the property $G^2=G$, is θ-WASS.

(e) If $(G,\cdot)$ is a H_v-group containing a scalar element, then all the maps $f: G\times G \longrightarrow G$ with $f(x,y)\in xy$ for all $x,y\in G$, are onto,

i.e. $f \in \Theta$. This is so because, if s is a scalar then $f(s,G)=G$.

Thus we obtain that $$\#\Theta = \prod_{(x,y)\in G^2} (\#xy) \ .$$

5.1.4 EXAMPLE

An example of the type of b-semigroups in the Remark 5.1.3(d) is defined as follows:

Fix an element $s \in G$. We define the product by setting

$$x \circ y = \begin{cases} s & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y \end{cases}$$

Then, $(\circ)$ is associative since

$$(x \circ y) \circ z = x \circ (y \circ z) = \begin{cases} x & \text{if} \quad x = y = z \\ s & \text{otherwise} \end{cases}$$

and, moreover, $G \circ G = G$.

5.2 H_v-HYPERGROUPOID ALGEBRA

5.2.1 DEFINITION

Let $(G,\cdot)$ be θ-WASS with $|G| = n$ and $\theta_\circ \in \Theta$ be associative. Let K be a field, and

$$K[G] = Z[G] \otimes_Z K$$

be the set of functions on G with values in K, i.e. K[G] is the set of formal linear combinations of elements of G with coefficients from K. In K[G] the ordinary addition (+) can be defined by setting

$$(f_1 + f_2)(g) = f_1(g) + f_2(g) \quad \text{for all} \quad g \in G \quad \text{and} \quad f_1, f_2 \in K[G].$$

Furthermore, consider the hyperproduct $(*)$, called **convolution**, defined for every f_1, f_2 of K[G] as follows

$$f_1 * f_2 = \{ f_\theta : f_\theta(g) = \sum_{\theta(x,y)=g} f_1(x)\, f_2(y), \quad \theta \in \Theta \}.$$

According to the above Remark 5.1.2, the defined convolution keeps the constant sum of n^2 terms for every $\theta \in \Theta$.

5.2.2 **THEOREM**

The $(K[G], +, *)$ is a H_v-ring which is, in fact, a multiplicative H_v-ring where the inclusion distributivity is valid.

Proof

Obviously, $(K[G], +)$ is a group.

Let $f_1, f_2, f_3 \in K[G]$. Then

$$(f_1 * f_2) * f_3 = \left\{ f_\theta : f_\theta(u) = \sum_{\theta(x,y)=u} f_1(x)\, f_2(y), \quad \theta \in \Theta \right\} * f_3 =$$

$$= \left\{ f_\varphi : f_\varphi(g) = \sum_{\varphi(u,z)=g} \sum_{\theta(x,y)=u} f_1(x)\, f_2(y)\, f_3(z), \ \theta, \varphi \in \Theta \right\} \ni f_{\theta_\circ}$$

where $f_{\theta_\circ}(g) = \sum_{\theta_\circ(u,z)=g} \sum_{\theta_\circ(x,y)=u} f_1(x)\, f_2(y)\, f_3(z).$

On the other hand

$$f_1 * (f_2 * f_3) =$$

$$\left\{ f_{\underline{\varphi}} : f_{\underline{\varphi}}(g) = \sum_{\underline{\varphi}(x,v)=g} \sum_{\underline{\theta}(y,z)=v} f_1(x)\, f_2(y)\, f_3(z), \quad \underline{\theta}, \underline{\varphi} \in \Theta \right\} \ni f_{\underline{\theta}_\circ}$$

where $f_{\underline{\theta}_\circ}(g) = \sum_{\underline{\theta}_\circ(x,v)=g} \sum_{\underline{\theta}_\circ(y,z)=v} f_1(x)\, f_2(y)\, f_3(z).$

But $\theta_\circ$ is associative, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$ for all x, y, z in G; which means that

$$\theta_o\left(\theta_o(x,y),z\right) = \underline{\theta}_o\left(x,\ \underline{\theta}_o(y,z)\right)$$

Therefore, $f_{\theta_o} = f_{\underline{\theta}_o}$. So

$$(f_1^* f_2)^* f_3 \cap f_1^* (f_2^* f_3) \neq \emptyset$$

Now, for the left distributivity, we have

$$f_1^* (f_2 + f_3) = \left\{f_\theta : f_\theta(g) = \sum_{\theta(x,y)=g} f_1(x)\left[(f_2 + f_3)(y)\right],\ \theta\in\Theta\right\} =$$

$$= \left\{f_\theta : f_\theta(g) = \sum_{\theta(x,y)=g} \left[f_1(x)\ f_2(y) + f_1(x)\ f_3(y)\right],\ \theta\in\Theta\right\}$$

and

$$f_1^* f_2 + f_1^* f_3 = \left\{f_\rho : f_\rho(g) = \sum_{\rho(x,y)=g} f_1(x)\ f_2(y),\ \rho\in\Theta\right\} +$$

$$+ \left\{f_\varphi : f_\varphi(g) = \sum_{\varphi(x,y)=g} f_1(x)\ f_3(y),\ \varphi\in\Theta\right\} =$$

$$= \left\{f_{\rho\varphi}\equiv f_\rho + f_\varphi :\ f_{\rho\varphi}(g) = \sum_{\rho(x,y)=g} f_1(x)\ f_2(y) + \sum_{\varphi(r,s)=g} f_1(r)\ f_3(s),\ \rho,\varphi\in\Theta\right\}$$

Therefore, $f_1^* (f_2 + f_3) \subseteq f_1^* f_2 + f_1^* f_3$ which is the left inclusion distributivity.

Similarly the right inclusion distributivity is valid. ∎

5.2.2 DEFINITION

The above H_v-ring is called **hypergroupoid H_v-algebra**.

5.2.3 OPEN PROBLEM

Find a sufficient and necessary condition to be valid by a hypergroupoid in order that a WASS convolution can be defined.

5.2.4 REMARK

Given a hypergroupoid $(G,\cdot)$ one can define a hypergroupoid H_v-algebra by "enlarging" the hyperoperation $(\cdot)$ as follows: Take any hyperoperation $(*)$ on G such that $(G,*)$ is Θ-WASS hypergroupoid. Then take a new hyperoperation $\square = \cdot \cup *$, i.e. $x \square y = (xy) \cup (x*y)$ for all x,y in G. obviously, on $(G,\square)$ the hypergroupoid algebra can be defined.

5.2.5 EXAMPLE

Consider the H_b-group $(Z_3,\oplus)$ where the b-group is $(Z_3,+)$ and the only non-singleton products are: $\bar{1}\oplus\bar{1}=\{\bar{0},\bar{1}\}$, $\bar{2}\oplus\bar{2}=\{\bar{0},\bar{1},\bar{2}\}$. The element $\bar{0}$ is scalar, so, from Remark 5.1.3(e), every map $\theta: Z_3^2 \longrightarrow Z_3$ with $\theta(x,y) \in xy$, is an element of Θ. Thus $|\Theta| = 2 \cdot 3 = 6$. More precisely, setting $\Theta = \{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6\}$ we have

$\theta(\bar{0},\bar{0})=\bar{0}$, $\theta(\bar{0},\bar{1})=\theta(\bar{1},\bar{0})=\bar{1}$, $\theta(\bar{0},\bar{2})=\theta(\bar{2},\bar{0})=\bar{2}$, $\theta(\bar{1},\bar{2})=\theta(\bar{2},\bar{1})=\bar{0}$,

for all $\theta \in \Theta$, and

$\theta_1(\bar{1},\bar{1})=\bar{0}$, $\theta_1(\bar{2},\bar{2})=\bar{0}$; $\theta_2(\bar{1},\bar{1})=\bar{0}$, $\theta_2(\bar{2},\bar{2})=\bar{1}$; $\theta_3(\bar{1},\bar{1})=\bar{0}$, $\theta_3(\bar{2},\bar{2})=\bar{2}$;

$\theta_4(\bar{1},\bar{1})=\bar{1}$, $\theta_4(\bar{2},\bar{2})=\bar{0}$; $\theta_5(\bar{1},\bar{1})=\bar{1}$, $\theta_5(\bar{2},\bar{2})=\bar{1}$; $\theta_6(\bar{1},\bar{1})=\bar{1}$, $\theta_6(\bar{2},\bar{2})=\bar{2}$.

Every hyperproduct of elements of $K[Z_3]$ has, at most, 6 elements. Let $r,s \in K[Z_3]$, then

$$r*s = \{ t_\theta : t_\theta(g) = \sum_{\theta(x,y)=g} r(x)\, s(y), \quad \theta \in \Theta \}.$$

For every $\theta \in \Theta$ on $t_\theta(g)$, $g \in Z_3$, 9 products of the form $r(x)s(y)$ are used. So, for example, we have

$t_{\theta_1}(\bar{0}) = r(\bar{0})s(\bar{0})+r(\bar{1})s(\bar{2})+r(\bar{2})s(\bar{1})+r(\bar{1})s(\bar{1})+r(\bar{2})s(\bar{2})$,

$t_{\theta_1}(\bar{1}) = r(\bar{0})s(\bar{1})+r(\bar{1})s(\bar{0})$, $t_{\theta_1}(\bar{2}) = r(\bar{0})s(\bar{2})+r(\bar{2})s(\bar{0})$;

$t_{\theta_2}(\bar{0}) = r(\bar{0})s(\bar{0})+r(\bar{1})s(\bar{2})+r(\bar{2})s(\bar{1})+r(\bar{1})s(\bar{1})$,

$t_{\theta_2}(\bar{1}) = r(\bar{0})s(\bar{1})+r(\bar{1})s(\bar{0})+r(\bar{2})s(\bar{2})$, $t_{\theta_2}(\bar{2}) = r(\bar{0})s(\bar{2})+r(\bar{2})s(\bar{0})$;

$t_{\theta_3}(\bar{0}) = r(\bar{0})s(\bar{0})+r(\bar{1})s(\bar{2})+r(\bar{2})s(\bar{1})+r(\bar{1})s(\bar{1})$,

$t_{\theta_3}(\bar{1}) = r(\bar{0})s(\bar{1})+r(\bar{1})s(\bar{0})$, $t_{\theta_3}(\bar{2}) = r(\bar{0})s(\bar{2})+r(\bar{2})s(\bar{0})+r(\bar{2})s(\bar{2})$;

and similar for the t_{θ_4}, t_{θ_5}, t_{θ_6}.

5.2.6 EXAMPLE

Consider the H_b-group $(\mathbf{Z}_{mn},\oplus)$ defined in Example 1.2.3. Then according to the Remark 5.1.3(e), the set θ consists of only two elements $\theta=\{\theta_1,\theta_2\}$ and $\theta(x,y)=x+y$ for all $\theta\in\theta$ and $(x,y)\neq(\bar{0},m)$ and $\theta_1(\bar{0},m)=\bar{0}$, $\theta_2(\bar{0},m)=m$.

The map θ_2 leads to the known convolution on $(\mathbf{Z}_{mn},+)$.

The map θ_1 gives in the expression $t_{\theta_1}(g) = \sum_{\theta_1(x,y)=g} r(x)\, s(y)$,

for every element different from $\bar{0}$ and m, a sum of mn elements.

Moreover, $t_{\theta_1}(\bar{0}) = \sum_{\theta_1(x,y)=\bar{0}} r(x)\, s(y)$ is a sum of $mn+1$ terms of form $r(x)s(y)$ and $t_{\theta_1}(m) = \sum_{\theta_1(x,y)=m} r(x)\, s(y)$ is a sum of $mn-1$ terms of $r(x)s(y)$.

Για να μη χάνουμε χρόνο
ανακαλύψαμε τα κομπιούτερς.
Τώρα χάνουμε το χρόνο
σ'αυτά.

So as not to lose time
we discovered the computer.
Now we waste our time
on it.

CHAPTER 6

GENERALIZED PERMUTATIONS

In the following two chapters we deal with representations of H_v-groups by generalized permutations and those chapters contain, mainly, an expanding version of the author's papers [36], [38] written for hypergroups.

6.1 THE GENARALIZED PERMUTATIONS

6.1.1 DEFINITIONS

Let X be a set. A map $f:X \longrightarrow \mathcal{P}(X)-\{\emptyset\}$ is called a **generalized permutation** of X if

$$\bigcup_{x\in X} f(x) = f(X) = X,$$

i.e. the reproductive axiom is valid. Recall that the associated to f relation $\bigcup_{x\in X} \{x\}\times f(X)$ on X has both domain and range X. Denote

M_X the set of generalized permutations on X. For a H_v-group $(X,\cdot)$ and $a\in X$, the generalized permutation defined by $f_a(x) = ax$ will be called a **left translation generalized permutation** or an **inner** generalized permutation.

By an **arrow** of f we mean a pair $(x,y)\in X^2$ with $y\in f(x)$.

We say that $f_2\in M_X$ **contains** $f_1\in M_X$ (or f_1 is a **subpermutation** of f_2), and we write $f_1 \subset f_2$, if $f_1(x) \subset f_2(x)$ for every x of X. If $f_1\neq f_2$ then we call f_1 a **proper** subpermutation of f_2 and we write $f_1 \subsetneq f_2$. The generalized permutation t with $t(x)=X$ for all $x\in X$, is called **universal** and contains all the elements of M_X. An $f\in M_X$ will be called **minimal** if it has no proper subpermutation. We will denote by $\underline{M}_X$ the set of all minimal elements of M_X. Obviously every ordinary permutation is minimal.

The **union** $f= \bigcup_{i\in I} f_i$ of a family $\{f_i : i\in I\}$ of generalized permutations, is defined by $f(x)= \bigcup_{i\in I} f_i(x)$ for all $x\in X$.

For $f \in M_X$ the **converse** of f, denoted $\bar{f}$, is the generalized permutation defined as follows

$$\bar{f}(x) = \{ z\in X : f(z) \ni x \} .$$

Therefore, we obtain $\bar{f}$ from f reversing the arrows.

We call $f\circ\bar{f}$ the generalized permutation **associated** to $f\in M_X$.

Remark that for every ordinary permutation the associated generalized permutation is the identity permutation.

6.1.2 REMARKS

(i) It is obvious that if $p,f_1,\dots,f_k,g_1,\dots,g_k \in M_X$ are such that $p\subset f_1\circ\dots\circ f_k$ and $f_1\subset g_1,\dots,f_k\subset g_k$ then $p\subset g_1\circ\dots\circ g_k$, where $\circ$ is the usual composition: if $f,g\in M_X$ then

$$f\circ g(x) = \bigcup_{y\in g(x)} f(y) \quad \text{for all } x\in X,$$

i.e. the composition of the associated relations written in reverse order. In other words, $(M_X,\circ)$ is a po-monoid.

(ii) Every map $f:X \longrightarrow \mathcal{P}(X)-\{\emptyset\}$ which contains a generalized permutation is itself a generalized permutation.

(iii) The intersection of two generalized permutations f_1, f_2 is the map $(f_1 \cap f_2)(x)=f_1(x) \cap f_2(x)$, for all $x \in X$. The intersection is not always a generalized permutation since the reproductive axiom is not necessarily valid.

(iv) Let $<$ be a total ordering on X^2. One applies a **deleting arrows procedure (dap)** on any $f \in M_X$ if, beginning from the minimal arrow, he deletes every arrow whenever the remaining map is a generalized permutation. It is immediate that, for any total ordering, the dap on any f gives a minimal generalized permutation. Therefore any f contains at least one element of M_X. Moreover if $p \in M_X$ with pcf one can find an ordering on X^2 such that applying the dap on f we obtain the p. It is enough in this ordering to have $(a,b)<(c,d)$, if $b \notin p(a)$ and $d \in p(c)$.

6.1.3 THEOREM

Let $f \in M_X$. Then $f \in M_X$ if and only if the following condition is valid: if $a \neq b$ and $f(a) \cap f(b) \neq \emptyset$, then $f(a)=f(b)$ and $f(a)$ is a singleton.

Proof

Let $f \in M_X$ and suppose that there are $a,b \in X$, $a \neq b$ such that

$f(a) \cap f(b) \neq \emptyset$, and that $f(a)$ is not a singleton.

Choose $u \in f(a) \cap f(b)$ and consider $f': X \longrightarrow \mathcal{P}(X)$, defined by $f'(a)=f(a)-\{u\}$ and $f'(x)=f(x)$ if $x \neq a$. Then $f' \in M_X$ and $f' \subsetneq f$ which is a contradiction.

Conversely, suppose that $f' \subsetneq f$, for some $f' \in M_X$, i.e. there exists $a \in X$ such that $f'(a) \subsetneq f(a)$. Choose $b \in f(a)-f'(a)$. Then there exists an element $d \neq a$ such that $b \in f'(d) \subset f(d)$. We have

$b \in f(a) \cap f(d)$ while $f(a) \supseteq b \cup f'(a)$, i.e. $|f(a)|>1$. ∎

6.1.4 COROLLARY

Let $f: X \longrightarrow \mathcal{P}(X)$. Then $f \in M_X$ if and only if there exist two partitions of X, $\{D_i\}_{i\in I}$ and $\{F_i\}_{i\in I}$, such that $|D_i|>1 \Rightarrow |F_i|=1$ for all $i\in I$, and $f= \bigcup_i D_i \times F_i$. Moreover, every such pair of partitions determines an element of M_X.

6.1.5 REMARKS

The above Theorem 6.1.3 gives an explicit description of M_X. It follows that if $f\in M_X$ then $\bar{f}\in M_X$. Moreover, for every x of X,

$$(f\circ\bar{f})(x) = f\{ u: f(u) \ni x \} = \bigcup_{f(u)\ni x} f(u)$$

Therefore,

$$(f\circ\bar{f})(x) = \{ y: \exists\, u \in X , \{x,y\} \subset f(u) \}.$$

So $I \subset f\circ\bar{f}$ for all $f\in M_X$ where I is the identity permutation.

6.1.6 EXAMPLE

Consider the P-hypergroup defined by the subset P of a group $(G,\cdot)$. Then for $a\in G$ the left translation L_a is $L_a(x) = aPx$.

The converse mapping $\bar{L}_a$ is defined by the relation

$$\bar{L}_a(x) = P^{-1}a^{-1}x, \quad \text{for all } x\in G.$$

The associated generalized permutation is defined by the relation

$$(L_a\circ\bar{L}_a)(x) = aPP^{-1}a^{-1}x \quad \text{for all } x\in G.$$

6.1.7 REMARK

Let $f\in M_X$ then the powers of the associated permutation are

$$(f\circ\bar{f})^2(x) = (f\circ\bar{f})\circ(f\circ\bar{f})(x) = \bigcup_{z_1\in f\circ\bar{f}(x)} \{z_2: \exists h_2\in X, \{z_1,z_2\}\subset f(h_2)\} =$$

$$= \{ z_2: \exists h_1,h_2\in X;\ z_1 ,\ \{x,z_1\}\subset f(h_1),\ \{z_1,z_2\}\subset f(h_2) \}$$

Therefore, for all $t>0$ we have

$$(f\circ\bar{f})^t(x) = \{\ y:\ \exists h_1,\ldots,h_t; z_o,z_1,\ldots,z_t\ ,\ \text{with}\ \ z_o=x, z_t=y,$$

$$\{z_{v-1},z_v\} \subset f(h_v)\ \text{for}\ v=1,\ldots,t\ \ \}$$

From this expression of the powers we remark that

$$\text{if}\ \ y \in (f\circ\bar{f})^t(x)\quad \text{then also}\quad x \in (f\circ\bar{f})^t(y).$$

We define the **height** of $f\in M_X$ to be the least integer r such that $(f\circ\bar{f})^{r+1} = (f\circ\bar{f})^r$. Since $f\circ\bar{f}$ is a non-decreasing function the height exists and is at most the cardinality of X. The ordinary permutations have height 1.

6.1.8 THEOREM

Let $(H,\cdot)$ be H_v-group and s be a single element, then the inner generalized permutation $L_s(x)=sx$ has height 1. More precisely $L_s\circ\bar{L}_s(x) = \beta^*(x)$ for all $x\in H$.

Proof

Using Theorem 1.3.3 we obtain for all $x\in H$

$$L_s\circ\bar{L}_s(x) = L_s(\{\ z\in H:\ L_s(z)\ni x\ \}) = L_s(\{\ z\in H:\ sz\ni x\ \}) =$$

$$= \bigcup_{sz\ni x} sz = \bigcup_{\beta^*(sz)\ni x} \beta^*(sz) = \beta^*(x).$$

So $L_s\circ\bar{L}_s(x) = \beta^*(x)$ for all $x\in H$. Therefore

$$(L_s\circ\bar{L}_s)^2(x) = L_s\circ\bar{L}_s(x) = \beta^*(x)\quad \text{for all}\ x\in H. \qquad \blacksquare$$

6.1.9 THEOREM

Let $(G,\cdot)$ be a group with $|G|=n$ and fix $x,y,z\in G$ such that $xy\neq z$. Consider the H_b-group $(G,\circ)$ defined by setting $x\circ y=\{xy,z\}$ and $g\circ g'=gg'$ for all $(g,g')\in G^2-\{(x,y)\}$. Then the height of every inner generalized permutation of $(G,\circ)$ is 1.

Proof

The only non ordinary inner generalized permutation is the L_x, i.e.

$$L_x(y) = \{xy,z\}\quad \text{and}\quad L_x(g_1)=xg_1\quad \text{for all}\ g_1\in G-\{y\},$$

then $\bar{L}_x(z) = \{y, x^{-1}z\}$ and $\bar{L}_x(g_1) = x^{-1}g_1$ for all $g_1 \in G-\{z\}$.

So $L_x \circ \bar{L}_x(z) = L_x(\{y, x^{-1}z\}) = \{xy, z\}$

$L_x \circ \bar{L}_x(xy) = L_x(y) = \{xy, z\}$ and

$L_x \circ \bar{L}_x(g_1) = L_x(x^{-1}g_1) = g_1$ for all $g_1 \in G-\{xy, z\}$.

From the above it is obtained that the height of L_x - and consequently the height of all inner generalized permutations - is 1. ■

Exercise

Generalize the result in the case where the above $L_x(y)$ has more than two elements.

6.1.10 EXAMPLE

In a similar direction one has some exercises like the following: Consider the H_b-group $(Z_5, \oplus)$ obtained from $(Z_5, +)$ where $\bar{2} \oplus \bar{0} = \{\bar{2}, \bar{3}\}$, $\bar{2} \oplus \bar{1} = \{\bar{3}, \bar{4}\}$ and for the rest sums we have $x \oplus y = x+y$. Then the only one non ordinary inner permutation is the L_2 where

$$L_2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \{\bar{2}, \bar{3}\} & \{\bar{3}, \bar{4}\} & \bar{4} & \bar{0} & \bar{1} \end{pmatrix} \text{ so } \bar{L}_2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{3} & \bar{4} & \bar{0} & \{\bar{0}, \bar{1}\} & \{\bar{1}, \bar{2}\} \end{pmatrix}$$

and $$L_2 \circ \bar{L}_2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{0} & \bar{1} & \{\bar{2}, \bar{3}\} & \{\bar{2}, \bar{3}, \bar{4}\} & \{\bar{3}, \bar{4}\} \end{pmatrix}$$

thus $$(L_2 \circ \bar{L}_2)^2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} \\ \bar{0} & \bar{1} & \{\bar{2}, \bar{3}, \bar{4}\} & \{\bar{2}, \bar{3}, \bar{4}\} & \{\bar{2}, \bar{3}, \bar{4}\} \end{pmatrix}.$$

Therefore, it is obtained that $\text{height} L_2 = 2$.

6.1.11 EXAMPLE

Consider the S-construction 1.5.10. Then for every $x \in S_1$, the inner generalized permutation L_x is defined as follows:

$$\left.\begin{aligned} L_x(y) &= xy \qquad \text{for } y\in S_i \\ L_x(z) &= S_i \cup S_j \quad \text{for all } z\in S_j,\ j\neq i \end{aligned}\right\}$$

Therefore,

$$\left.\begin{aligned} \bar{L}_x(u) &= \Big(\bigcup_{j\neq i} S_j\Big) \cup \{\, u=xy \mid y\in S_i \,\} \quad \text{if } u\in S_i \\ \bar{L}_x(v) &= S_j \quad \text{if } v\in S_j \neq S_i \ , \end{aligned}\right\}$$

from which it is obtained

$$\left.\begin{aligned} L_x\circ\bar{L}_x(u) &= S \qquad \text{for } u\in S_i \\ L_x\circ\bar{L}_x(v) &= L_x(S_j) = S_i\cup S_j \quad \text{for } v\in S_j\neq S_i \ . \end{aligned}\right\}$$

So $(L_x\circ\bar{L}_x)^2(w) = S = (L_x\circ\bar{L}_x)^3(w)$ for all $w\in S$; thus $\text{height} L_x = 2$.

6.2 THE CONDITION θ

6.2.1 DEFINITION

A generalized permutation f is said to satisfy the **condition θ** if

$$\text{for all } x\in X,\ z\in f(x) \text{ it implies } f(z) = f(x) \ .$$

Denote the set of all generalized permutations that satisfies the condition θ by M_θ. The identity and the universal belongs to M_θ. Using the Remark 6.1.7 it is easy to show that every $(f\circ\bar{f})^r$ is in M_θ, where r is the height of $f\in M_X$.

6.2.2 THEOREM

(i) $f\in M_\theta$ implies $x\in f(x)$ for all x. Thus $f\supset I$ if $f\in M_\theta$.

(ii) $f\in M_\theta$ if and only if $f(x) = \{\, z: f(z)=f(x) \,\}$ for all x.

(iii) $f\in M_\theta$ implies $f=\bar{f}$.

Proof

If $f\in M_\theta$, then $x\in f(y)$ implies $f(x)=f(y)$, so $x\in f(x)$ and thus Icf. The rest of the proof is obvious. ∎

From this theorem one remarks that $f\in M_\theta$ if and only if the associated relation $\bigcup_{x\in X} \{x\}\times f(X)$ is an equivalence relation.

6.2.3 DEFINITION

The θ relation is defined as follows:
For $f\in M_\theta$ let θ denote the kernel of f, i.e. $x\theta y$ iff $f(x)=f(y)$, and let $\{X_s\}_{s\in S}$ denote the partition of equivalence classes of θ.
Then $f(x)=X_s$ for all $x\in X_s$ by Theorem 6.2.2(iii).

6.2.4 THEOREM

Let $(X,\cdot)$ be a finite hypergroup and consider the ℓ_x-fundamental relation defined on X by $x\in X$. Then ℓ_x coincides with the kernel of $(f_x\circ\bar{f}_x)^r$ where f_x is the left translation by x, and r is the height of f_x.

Proof
Immediate from Remark 6.1.7. ∎

The above theorem gives a method to compute the ℓ_x fundamental equivalence class of any element a of X, i.e.

$$\ell_x(a) = (f_x\circ\bar{f}_x)^r(a).$$

The family $\{X_s\}_{s\in S}$ becomes a hypergroup, the "fundamental" one defined by the element x. Obviously we have analogous results if we use the γ_x fundamental relations and the right translations.

6.2.5 THEOREM

If $f\in M_X$ then $(\bar{f}\circ f)(x) = \{ z\in X: f(z)=f(x) \}$

Proof
Follows from Corollary 6.1.4. ∎
Remark that for all $f\in M_X$ we have $\bar{f}\circ f, f\circ\bar{f} \in M_\theta$.

6.2.6 THEOREM

If $f,f' \in M_\theta$ then $f \wedge f' \in M_\theta$.

Proof
Follows from Theorem 6.2.2(i). ∎

6.2.7 DEFINITION

A generalized permutation g will be called **partial left (resp. right) identity** of f if $g \circ f > f$ (resp. $f \circ g > f$). If we have $g \circ f = f$ (resp. $f \circ g = f$) then g is called **scalar partial left (resp. right) identity** of f.

6.2.8 THEOREM

Let $f \in M_\theta$. Then a generalized permutation g is a scalar partial left identity of f if and only if $g \in f$. We have a similar result for right partial identities.

Proof
Although the proof follows from Theorem 6.2.2, here we present a proof more familiar to hypergroupists.

From Definition 6.2.3, if we take the family of indices $\{J_s\}_{s \in S}$ such that we have the bijections $J_s \longrightarrow X_s : j \longrightarrow x_{sj}$ then we have

$$X_s = \{ x_{sj} : j \in J_s \} \quad \text{and, moreover,} \quad f(x_{sj}) = X_s \text{ for all } j \in J_s .$$

Let $g \in f$ and X_s a partition class. Then we must have

$$\bigcup_{j \in J_s} g(x_{sj}) = X_s .$$

So $$g \circ f(x_{sj}) = g(f(x_{sj})) = g(X_s) = X_s = f(x_{sj}) \quad \text{for all} \quad j \in J_s .$$

Therefore, for all X_s's we obtain $g \circ f = f$.

On the other hand

$$f \circ g(x_{sj}) = \bigcup_{y \in g(x_{sj})} f(y) = \bigcup_{y \in g(x_{sj})} X_s = X_s = f(x_{sj}) \quad \text{for all } j \in J_s .$$

Therefore, every $g \in f$ is a scalar partial identity of f.

Conversely, let $g \in M_X$ such that $g \circ f = f$, then we have for all $x_{sj} \in X$,

$$g \circ f(x_{sj}) = g(X_s) = X_s = f(x_{sj}) \quad \text{for all} \quad s \in S.$$

That means that $g \subset f$.

On the other hand let $f \circ g = f$ then

$$f \circ g(x_{sj}) = \bigcup_{y \in g(x_{sj})} f(y) = X_s = f(x_{sj})$$

therefore, $y \in X_s$, i.e. $g(x_{sj}) \subset X_s$ for all $s \in S$, so $g \subset f$. ∎

From the above theorem and the Definition 6.2.7 we obtain that every f of M_θ is an idempotent. Moreover, from the proof of the above theorem we obtain that if $g_1, g_2 \subset f$ then $g_1 \circ g_2 \subset f$. We also have $f \circ \bar{f} = f \circ f = f$ so the height of every $f \in M_\theta$ is equal to 1.

6.2.9. THEOREM

Let $f \in M_\theta$ and $M_f = \{ g \in M_X : g \subset f \}$. Then M_f is a hypergroup with respect to the hyperoperation $(*)$ defined by

$$f_1 * f_2 = \{ p \in M_X : p \subset f_1 \circ f_2 \} \quad \text{for all} \quad f_1, f_2 \in M_f .$$

Proof

For every $f_1, f_2, f_3 \in M_f$ we have

$$f_1 * (f_2 * f_3) = \{ g \in M_X : g \subset f_1 \circ f_2 \circ f_3 \} = (f_1 * f_2) * f_3 \qquad (I)$$

The reproductivity is also valid. ∎

6.2.10 REMARKS

(i) The above relation (I) implies that if $f_1, \ldots, f_s \in M_f$ then $f_1 * \ldots * f_s = \{ g \in M_X : g \subset f_1 \circ \ldots \circ f_s \}$

(ii) One also obtains that $g * f = f * g = M_f$ for all $g \in M_f$.

(iii) The hypergroup $(M_f, *)$ is a regular hypergroup since the identity permutation I is an element of M_f , and for every $g \in M_f$ we have $g * f = M_f = f * g \ni I$ (see [12]) .

6.2.11 EXAMPLE

Consider the H_b-group in Theorem 6.1.9. The only non ordinary inner generalized permutation is the L_x with height 1. In other words from Definition 6.2.1, we have $L_x \circ \bar{L}_x \in M_\theta$ where $xy \neq z$ and

$L_x \circ \bar{L}_x(z) = L_x \circ \bar{L}_x(xy) = \{xy,z\}$ and

$L_x \circ \bar{L}_x(g_i) = g_i$ for all $g_i \in G-\{xy,z\}$

In this case the set

$$M_{L_x} = \{ f : f \subset L_x \circ \bar{L}_x \}$$

contains 7 elements which are defined to be the identity for all elements except for the elements z and xy. The elements of M_{L_x} have for $i=1,\ldots,7$ in the pair $(f_i(z), f_i(xy))$, respectively the pairs (z,xy), (xy,z), $(z,\{xy,z\})$, $(xy,\{xy,z\})$, $(\{xy,z\},z)$, $(\{xy,z\},xy)$, and $(\{xy,z\},\{xy,z\})$. The first two of the above generalized permutations are the minimal ones. In the set M_f the hyperoperation (*) is defined as in Theorem 6.2.9 and $(M_{L_x},*)$ is a hypergroup with seven elements.

6.2.12 THEOREM

For every $f \in M_\theta$ the hypergroup $(M_f,*)$ is isomorphic to the cartesian product of the induced hypergroups defined by the universal permutations on the θ-classes of f.

Proof

Let $\{X_s\}_{s\in S}$ be the family of the θ-classes defined by the f, and let t_s be the universal permutation defined on X_s for every $s \in S$. Then the induced hypergroups are the ones defined, for every $s \in S$, on the sets $M_{t_s} = \{ g \in M_{X_s} : g \subset t_s \}$ using the (*) hyperoperation.

But since t_s is universal we have $M_{t_s} = M_{X_s}$.

Let us take the cartesian product $\times_{s\in S} M_{X_s}$ of the family $\{M_{X_s}\}_{s\in S}$.

Consider the map $T: M_f \longrightarrow \times_{s\in S} M_{X_s} : g \longrightarrow T(g) = \{g_s\}_{s\in S}$ where g_s is the restriction of g to X_s.

Since $\bigcup_{y\in X_s} g(y) = X_s$, $g_s \in M_{X_s}$ $\forall\, s\in S$. Clearly, T is injective.

Moreover, for $\{g_s\}_{s\in S}$ of $\underset{s\in S}{\times} M_{X_s}$, $g = \bigcup_{s\in S} g_s$ is a generalized permutation on X and $T(g) = \{g_s\}_{s\in S}$.

Now let $g, g' \in M_f$ then $g\circ g' \in M_f$ and, for every s and $y\in X_s$, the g_s, g'_s have $g\circ g'(y) = g\circ g'_s(y) = g_s(g'_s(y)) = g_s\circ g'_s(y)$.

This means that $(g\circ g')_s = g_s\circ g'_s \in M_{X_s}$. So $T(g\circ g') = \{g_s\circ g'_s\}_{s\in S}$.

Therefore, from Theorem 6.2.9 it is implied $T(g^*g') = \{g_s{}^*g'_s\}_{s\in S}$.

So $(M_f, {}^*) \cong (\underset{s\in S}{\times} M_{X_s}, {}^*)$ for all $f\in M_\theta$. ∎

Using Theorem 6.2.5 one obtains the following:

6.2.13 THEOREM

Let $f\in M_X$. Then $f\circ\bar{f}$ is a scalar partial left identity of f and contains all the scalar partial left identities of f.

From this theorem one observes that $f\circ\bar{f}$ is the greatest element of the set of all scalar partial identities of f. The identity permutation is a minimal scalar partial left identity for f, but there will be others. The $(\bar{f}\circ f)\cap(f\circ\bar{f})$ contains all scalar partial double identities of f as well as all scalar partial double identities of $\bar{f}$. So the partial double identities of f and $\bar{f}$ coincide.

Κατέχραψα στον κομπιούτερ
τις γνώσεις μου
κ'η σκέψη μου κρύφτηκε
στη διαδικασία.

I recorded all my knowledge
on the computer,
and my thought got lost
in the process.

CHAPTER 7

REPRESENTATIONS BY GENERALIZED PERMUTATIONS

7.1 A NON STANDARD HYPEROPERATION

It is an open problem to find one standard hypercomposition to represent H_v-groups by generalized permutations. The problem arises since in the definition of a hyperoperation $(\cdot)$ on a set X one has two steps,

first: $(\cdot)$ is a map $\cdot: X\times X \longrightarrow \mathcal{P}(X): (a,b) \longmapsto abcX$ and

second: $(\cdot)$ is a map

$$\cdot: \mathcal{P}(X)\times\mathcal{P}(X) \longrightarrow \mathcal{P}(X): (A,B) \longmapsto A\cdot B = \bigcup_{a\in A,\, b\in B} ab$$

On the other side, the law of composition of generalized permutations is, actually, an operation, thus it follows only the second step.

7.1.1 DEFINITION

Let $F = \{f_i: i=1,\dots,n\}$ be a set of n generalized permutations on X, $|X|=n$, not necessarily different from each other. A hyperoperation (*) in F is defined as follows

$$f_i * f_j = \{ f_s: x_s \in f_i(x_j) \}$$

The hyperoperation (*) depends on the choice of the set F, thus it is a non standard hyperoperation. Sometimes (*) coincides with the ordinary composition of permutations for example if F is a set of left inner generalized permutations. In this paragraph the (*) is presented but first we give the following definition.

7.1.2 DEFINITION

Let $(H,\cdot)$ be H_v-group and M_X be the set of generalized permutations in which a hyperoperation $(\circ)$ is defined. A map

$$T: H \longrightarrow M_X : h \longmapsto T(h) \qquad \text{such that}$$

$$T(h_1h_2) \cap T(h_1)\circ T(h_2) \neq \emptyset \quad \text{for all } h_1,h_2 \text{ in } H$$

is called a **representation of H by generalized permutations**. If

$$T(h_1h_2) \subseteq T(h_1)\circ T(h_2) \quad \text{for all } h_1,h_2 \text{ in } H, \quad \text{then T is called}$$

inclusion representation. Finally, if the **good condition**

$$T(h_1h_2) = \{ T(h): h\in h_1h_2 \} = T(h_1)\circ T(h_2) \quad \text{for all } h_1,h_2 \text{ in } H$$

is valid and T is 1-1, then T is called **faithful** representation.

If the good condition is valid then, obviously, an induced representation T^* for the hypergroup algebra is obtained.

7.1.3 THEOREM

Let $(H,\cdot)$ with $|H|=n$, be H_v-group. Consider the set of all inner generalized permutations in H

$$F = \{ f_i: f_i(h_j) = h_ih_j,\ i=1,\dots,n \}$$

Then the onto map $\pi : H \longrightarrow F : h_i \longmapsto f_i$ is a representation of H on the H_v-group $(F,*)$. Moreover, if the right cancellation law is valid in H, then π is faithful.

Proof
For every f_i, f_j, f_k of F, we have

$$f_i * (f_j * f_k) = \{f_v : h_v \in h_i(h_j h_k)\} \quad \text{and} \quad (f_i * f_j) * f_k = \{f_v : h_v \in (h_i h_j)h_k\}$$

so $$f_i * (f_j * f_k) \cap (f_i * f_j) * f_k \neq \emptyset .$$

Moreover,

$$f_i * F = \bigcup_{f_j \in F} f_i * f_j = \bigcup_{f_j \in F} \{ f_s : h_s \in f_i(h_j) \} = \{ f_s : h_s \in \bigcup_{h_j \in H} f_i(h_j) =$$

$$= \{ f_s : h_s \in h_i H \} = \{ f_s : h_s \in H \} = F$$

Similarly, $F * f_i = F$ for all $f_i \in F$. Therefore, $(F, *)$ is a H_v-group.
For every $h_i, h_j \in H$ we have

$$\pi(h_i h_j) = \{ \pi(h_k) : h_k \in h_i h_j \} = \{ f_k : h_k \in h_i h_j \} =$$

$$= \{ f_k : h_k \in f_i(h_j) \} = f_i * f_j = \pi(h_i) * \pi(h_j),$$

so π is a strong homomorphism, therefore π is a representation.
Now let us suppose that the right cancellation law is valid in H.
If $h_i \neq h_j$ and $\pi(h_i) = \pi(h_j)$ then we have for all $h_k \in H$, $f_i(h_k) = f_j(h_k)$ or $h_i h_k = h_j h_k$ or $h_i = h_j$ which is a contradiction.
So π is an injection. ■

7.1.4 EXAMPLE

Consider the set $Z_6 = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5} \}$ and consider the COW and WASS H_v-group $(Z_6, \oplus)$, where $(\oplus)$ is obtained from $(+)$ by enlarging only the following hypersums

$$\bar{1} \oplus \bar{2} = \{\bar{1}, \bar{3}\}, \quad \bar{2} \oplus \bar{4} = \{\bar{0}, \bar{4}\} \quad \text{and} \quad \bar{3} \oplus \bar{2} = \{\bar{3}, \bar{5}\}$$

Remark that the results were enlarged with elements belonging to the same class of Z_6 divided by $\{\bar{0}, \bar{2}, \bar{4}\}$.
The n inner generalized permutations corresponding to the elements $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$, are

$$f_0 = \mathrm{id}, \quad f_1 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{1} & \bar{2} & \{\bar{1},\bar{3}\} & \bar{4} & \bar{5} & \bar{0} \end{pmatrix}, \quad f_2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{2} & \bar{3} & \bar{4} & \bar{5} & \{\bar{0},\bar{4}\} & \bar{1} \end{pmatrix}$$

$$f_3 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{3} & \bar{4} & \{\bar{3},\bar{5}\} & \bar{0} & \bar{1} & \bar{2} \end{pmatrix}, \quad f_4 = \begin{pmatrix} \bar{1} \\ \overline{1+4} \end{pmatrix}, \quad f_6 = \begin{pmatrix} \bar{1} \\ \overline{1+5} \end{pmatrix}$$

Suppose we take the hypersums $(\bar{1}\oplus\bar{2})\oplus\bar{2} = \{\bar{1},\bar{3},\bar{5}\}$, $\bar{1}\oplus(\bar{2}\oplus\bar{2}) = \{\bar{5}\}$; they have $(\bar{1}\oplus\bar{2})\oplus\bar{2} \cap \bar{1}\oplus(\bar{2}\oplus\bar{2}) \neq \emptyset$.

For the corresponding inner generalized permutations we obtain in a similar way

$$(f_1 * f_2) * f_2 = \{f_1, f_3\} * f_2 = \{f_1, f_3, f_6\} \quad \text{and}$$

$$f_1 * (f_2 * f_2) = f_1 * f_4 = f_5$$

Moreover, remark that the generalized permutation

$$(f_1 \circ f_2) \circ f_2 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{5} & \bar{0} & \{\bar{1},\bar{5}\} & \bar{2} & \{\bar{1},\bar{3},\bar{5}\} & \bar{4} \end{pmatrix} = f_1 \circ (f_2 \circ f_2),$$

contains the following 10 elements of $M_{\mathbb{Z}_6}$

$$g_1 = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \bar{5} & \bar{0} & g_1(\bar{2}) & \bar{2} & g_1(\bar{4}) & \bar{4} \end{pmatrix} \quad \text{where}$$

$g_1(\bar{2})=\bar{1}$, $g_1(\bar{4})=\bar{3}$, $g_2(\bar{2})=\bar{1}$, $g_2(\bar{4})=\{\bar{1},\bar{3}\}$, $g_3(\bar{2})=\bar{1}$, $g_3(\bar{4})=\{\bar{3},\bar{5}\}$

$g_4(\bar{2})=\bar{1}$, $g_4(\bar{4})=\{\bar{1},\bar{3},\bar{5}\}$, $g_5(\bar{2})=\bar{5}$, $g_5(\bar{4})=\{\bar{1},\bar{3}\}$, $g_6(\bar{2})=\bar{5}$, $g_6(\bar{4})=\{\bar{1},\bar{3},\bar{5}\}$

$g_7(\bar{2})=\{\bar{1},\bar{5}\}$, $g_7(\bar{4})=\bar{3}$, $g_8(\bar{2})=\{\bar{1},\bar{5}\}$, $g_8(\bar{4})=\{\bar{1},\bar{3}\}$,

$g_9(\bar{2})=\{\bar{1},\bar{5}\}$, $g_9(\bar{4})=\{\bar{3},\bar{5}\}$ and $g_{10}(\bar{2})=\{\bar{1},\bar{5}\}$, $g_{10}(\bar{4})=\{\bar{1},\bar{3},\bar{5}\}$

from which the g_1 and g_5 are the minimal ones.

7.2 H_v-GROUP PERMUTATION REPRESENTATIONS

This paragraph focuses on the representation of hypergroups by generalized permutations. Theorem 6.2.9 leads to a hyperoperation which can represent some classes of hypergroups.

7.2.1 THEOREM

Let $(H,\cdot)$ be a finite hypergroup and L_a the left translation by a. We consider the hypergroup $(M_H,^*)$ where $(^*)$ is defined on M_H as in Theorem 6.2.9 using the universal generalized permutation, i.e.

$$f_1^*f_2 = \{\ p\in M_H:\ pcf_1\circ f_2\ \}\quad \text{for all}\quad f_1,f_2\in M_H\ .$$

Then the map

$$L : H \longrightarrow M_H : a \longmapsto L(a)=L_a$$

is a representation (not necessarily faithful).

Proof

Let $a,b\in H$. Then $L(ab)=\{\ L_c:\ c\in ab\ \}$. On the other hand

$$L(a)^*L(b) = L_a{}^*L_b = \{\ L_d:\ L_d\subset L_a\circ L_b\ \}.$$

But for all $x\in H$ and $L_c\in L(ab)$ we have $L_c(x)=cx\subset abx = L_a\circ L_b(x)$, so $L_c\in L(a)^*L(b)$, thus $L(ab)\subset L(a)^*L(b)$. Therefore L is an homomorphic representation. ∎

The above hypergroup $(M_H,^*)$ is in fact the one defined in [12].

This representation is not 1-1 except for some special cases. For example, if H has a right scalar element then L is faithful. However, we can obtain a faithful representation as in the following.

7.2.2 DEFINITION

An **extension** $(H_e,\cdot)$ of a hypergroup $(H,\cdot)$ is defined as follows:

If $(H,\cdot)$ has a scalar identity e, put $(H_e,\cdot)=(H,\cdot)$. If it does not, put $H_e:=H\cup\{e\}$ (where $e\notin H$) and extend the multiplication by setting $eh=he:=\{h\}$ for all $h\in H_e$.

It is immediate that $(H_e,\cdot)$ is a semihypergroup.

7.2.3 THEOREM

Let $(H,\cdot)$ be a hypergroup and $X=H_e$. Then the map

$$L: H_e \longrightarrow M_X: a \longmapsto L_a$$

is a faithful (homomorphic) representation of $(H_e,\cdot)$ in the semihypergroup $(M_X,^*)$.

Proof
Theorem 7.2.1 implies that L is a representation. Moreover, if $a\neq b$ then $L_a(e)=a$, $L_b(e)=b$ so $L_a\neq L_b$. Thus L is faithful. ∎

The above L is called an **extended regular representation** on H_e.

In the following fix a hypergroup $(H,\cdot)$ with $|H|=n$.

7.2.4 DEFINITION

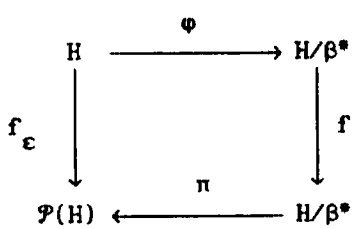
A generalized permutation f_ε on H is said to be **induced by a** permutation f of the set of the fudamental group H/β^*, if

$$f_\varepsilon(x) = f(\beta^*(x)) \quad \text{for all } x\in H.$$

where $f(\beta^*(x))$ denotes not one class but the set of the elements of the corresponding class.

7.2.5 REMARKS

(i) f_ε may be described by the diagram



where φ be the canonical map and π is such that

$$\pi(\beta^*(x))=\beta^*(x) \quad \text{for all } x\in H.$$

Therefore, we have

$$f_\varepsilon(x) = \pi\circ f\circ\varphi(x) = \pi\circ f(\beta^*(x)) = f(\beta^*(x)) .$$

(ii) Since the fundamental property is valid on H/β^*, one has

$f_\varepsilon(xy) = f(\beta^*(xy)) = f(\beta^*(z))$ for all $x,y\in H$ and for all $z\in xy$;

thus $f_\varepsilon(\beta^*(x)) = \bigcup_{y\in\beta^*(x)} f_\varepsilon(y) = \bigcup_{y\in\beta^*(x)} f(\beta^*(y)) = f(\beta^*(x))$.

So if f,g are two permutations on the set H/β^* then

$$f_\varepsilon \circ g_\varepsilon(x) = f_\varepsilon(g(\beta^*(x))) = (f\circ g)(\beta^*(x)) \quad \text{for all} \quad x\in H.$$

(iii) The induced generalized permutations f_ε can be defined for H_v-groups as well. The above remarks are also applied to H_v-groups in an obvious way. Finally, remark that the induced generalized permutation f_ε is the same for every H_v-group defined using a greater hyperoperation which has the same fundamental group.

7.2.6 EXAMPLE

Consider the very thin H_b-group defined in the Example 1.2.3. Fix the representatives $k_0,\dots k_{m-1}$ of the β^* classes, then every permutation $f: \mathbb{Z}_m/\beta^* \longrightarrow \mathbb{Z}_m/\beta^*$ gives a permutation of the set $\{ k_0,\dots,k_{m-1} \}$ and let us denote this permutation by the same symbol f. Then the induced by f generalized permutation is defined as follows

$$f_\varepsilon(x) = f(\beta^*(x)) = f(x+m\mathbb{Z}) = f(x)+m\mathbb{Z}.$$

7.2.7 THEOREM

Let $\{f_u : u\in U\}$ be a group of permutations on H/β^* and $\{f_{u\varepsilon} : u\in U\}$ be the set of induced generalized permutations on H. Consider the set

$$M_U = \{ g \in M_H : g \subset f_{u\varepsilon} \text{ for some } u \in U \}$$

and define the hyperoperation $(*)$ on M_U by setting

$$g_1 * g_2 = \{ p : p \subset g_1\circ g_2 \}$$

Then $(M_U, *)$ is a hypergroup.

Proof
Remark that $\{ f_{u\varepsilon} : u\in U \}$ is also a group with respect to the composition of functions. Moreover, if $g_1 \subset f_{u_1\varepsilon}$, $g_2 \subset f_{u_2\varepsilon}$ then $p \subset g_1 \circ g_2 \subset f_{u_1\varepsilon} \circ f_{u_2\varepsilon}$ so $p\in M_U$, meaning that $*$ is well defined.

For every $g_1, g_2, g_3 \in M_U$,

$$g_1 * (g_2 * g_3) = g_1 * \{ p \subset g_2 \circ g_3 \} = \bigcup_{p \subset g_2 \circ g_3} g_1 * p =$$

$$= \bigcup_{p \subset g_2 \circ g_3} \{ q: q \subset g_1 \circ p \} = \{ q: q \subset g_1 \circ g_2 \circ g_3 \} .$$

With a similar proof we finally have

$$g_1 * (g_2 * g_3) = \{ q : q \subset g_1 \circ g_2 \circ g_3 \} = (g_1 * g_2) * g_3 .$$

Therefore, $(M_U, *)$ is a semihypergroup.

In order to prove the reproductivity one needs the following:

LEMMA
If $u,v\in U$ and $g \subset f_{u\varepsilon}$, then $g \circ f_{v\varepsilon} = f_{u\varepsilon} \circ f_{v\varepsilon}$.

Proof
For $x \in H$ and $y \in \beta^*(x)$,

$$g \circ f_{v\varepsilon}(y) = g(f_v(\beta^*(x))) = g(\beta^*(x')) \quad \text{where} \quad x' \in f_v(\beta^*(x)).$$

Since $g \subset f_{u\varepsilon}$ and $g\in M_H$,

$$g(\beta^*(x')) = f_{u\varepsilon}(x') = f_{u\varepsilon}(\beta^*(x')) = f_{u\varepsilon}(f_v(\beta^*(x)))$$

which gives $g \circ f_{v\varepsilon} = f_{u\varepsilon} \circ f_{v\varepsilon}$.

Now we return to the proof of the theorem.

Let $g \in M_U$ and $g \subset f_{v\varepsilon}$, then

$$g^*M_U = \bigcup_{q\in M_U} g^*q = \bigcup_{q\in M_U} \{\, t:\ t\subset g\circ q \,\} = \bigcup_{q\subset f_{v\varepsilon},\, v\in U} \{\, t:\ t\subset g\circ q \,\} =$$

$$= \bigcup_{v\in U} \{\, t:\ t\subset g\circ f_{v\varepsilon} \,\} .$$

Thus, using the above lemma, we have

$$g^*M_U = \bigcup_{v\in U} \{\, t:\ t\subset f_{u\varepsilon}\circ f_{v\varepsilon} \,\} = \{\, t:\ t\subset f_{w\varepsilon} \text{ where } w\in U \,\} = M_U$$

because $\{ f_{u\varepsilon}:\ u\in U \}$ is a group so $\{ f_{u\varepsilon}\circ f_{v\varepsilon}:\ v\in U \} = \{ f_{u\varepsilon}: u\in U\}$.

Therefore, $(M_U, ^*)$ is a hypergroup. ■

Remark If $\{ f_{u\varepsilon}:\ u\in U \} = \{id\}$ and if we take the partition of H obtained by the fundamental equivalence β^*, then the hypergroup $(M_U, ^*)$ coincides with the hypergroup defined in Theorem 6.2.9.

7.2.8 THEOREM

Let $\{ L_a:\ a\in H \}$ be the group of left translations on H/β^* induced by the elements of H. Then the map

$$L : H \longrightarrow M_U : a \longmapsto L_{a\varepsilon}$$

is a representation of the hypergroup H.

Proof

We remark that $L_y = L_a$, $L(y) = L_{a\varepsilon}$ for all $y\in F_H(a)$. So for all $a\in H$,

$$L_{a\varepsilon}(x) = L_a(\beta^*(x)) = \beta^*(a)\cdot\beta^*(x) .$$

Consequently, for every $a,b,x \in H$,

$$L(ab) = L_{c\varepsilon} \quad \text{for all} \quad c \in ab \subset \beta^*(ab) = \beta^*(c)$$

and so $L(ab)(x) = L_{c\varepsilon}(x) = L_c(\beta^*(x)) = \beta^*(c)\cdot\beta^*(x)$

where $c\in ab$ and $\beta^*(c) = \beta^*(a)\cdot\beta^*(b)$.

On the other hand, $L(a)^*L(b) = L_{a\varepsilon}{}^*L_{b\varepsilon} = \{ p:\ p\subset L_{a\varepsilon}\circ L_{b\varepsilon} \}$

and if we take the element $L_{a\varepsilon}\circ L_b$ of $L_{a\varepsilon}{}^*L_{b\varepsilon}$ we obtain

$$L_{a\varepsilon}\circ L_{b\varepsilon}(x) = L_{a\varepsilon}(L_b(\beta^*(x))) = L_{a\varepsilon}(\beta^*(b)\cdot\beta^*(x)) = \beta^*(a)\cdot\beta^*(b)\cdot\beta^*(x).$$

Therefore, $L(ab) \subset L(a)^*L(b)$.

So L is a representation. ∎

7.2.9 EXAMPLE

Consider the special case of the Example 7.2.6 where $m\cdot n=3\cdot 2$ for which we have $\bar{0}\oplus\bar{3} = \{\bar{0},\bar{3}\}$ and $x\circledast y=x+y$ for all $(x,y)\in Z_6^2-\{(\bar{0},\bar{3})\}$. So $Z_6/\beta^* \cong Z_3$ and the three β^* classes are

$$\beta^*(\bar{0})=\{\bar{0},\bar{3}\},\quad \beta^*(\bar{1})=\{\bar{1},\bar{4}\},\quad \beta^*(\bar{2})=\{\bar{2},\bar{5}\}.$$

Consider the set $\{f_{\bar{0}},f_{\bar{1}},f_{\bar{2}}\}$ of permutations on the Z_6/β^* such that

$$f_{\bar{0}} = \mathrm{id}\ ,\quad f_{\bar{1}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} \\ \bar{1} & \bar{2} & \bar{0} \end{pmatrix},\quad f_{\bar{2}} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} \\ \bar{2} & \bar{0} & \bar{1} \end{pmatrix}$$

Then the corresponding induced generalized permutations for the set $U=\{\bar{0},\bar{1},\bar{2}\}$, are

$$f_{\bar{0}\varepsilon} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \{\bar{0},\bar{3}\} & \{\bar{1},\bar{4}\} & \{\bar{2},\bar{5}\} & \{\bar{0},\bar{3}\} & \{\bar{1},\bar{4}\} & \{\bar{2},\bar{5}\} \end{pmatrix}$$

$$f_{\bar{1}\varepsilon} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \{\bar{1},\bar{4}\} & \{\bar{2},\bar{5}\} & \{\bar{0},\bar{3}\} & \{\bar{1},\bar{4}\} & \{\bar{2},\bar{5}\} & \{\bar{0},\bar{3}\} \end{pmatrix}$$

$$f_{\bar{2}\varepsilon} = \begin{pmatrix} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \\ \{\bar{2},\bar{5}\} & \{\bar{0},\bar{3}\} & \{\bar{1},\bar{4}\} & \{\bar{2},\bar{5}\} & \{\bar{0},\bar{3}\} & \{\bar{1},\bar{4}\} \end{pmatrix}$$

Each of the above induced generalized permutations contains 7^3 generalized permutations. Therefore, the set

$$M_U = \{\, g \in M_H : g \subset f_{u\varepsilon} \text{ for some } u \in U \,\}$$

has $3\cdot 7^3$ elements.

Remark

One can see that the associated to L_x generalized permutation $L_x \circ \bar{L}_x$, defined in Example 6.2.11, contains seven elements. The hypergroup defined in M_{L_x} can be represented faithfully using hypergroups of the above form or of the Example 7.2.6. Actually, each induced generalized permutation of the Example 7.2.9 is a cartesian product, three times, of the set $L_x \circ \bar{L}_x$. This remark leads to the following:

Exercise

Give an explicit description the H_v-groups which can be represended faithfully using the general case $m \cdot n$ of the Example 7.2.6.

Χωρίς ρυθμό η ματιά μου
γίνεται ευθεία.
Ευθεία διδιάστατη που κρύβει
τους τριδιάστατους παλμούς.

Without rhythm, my glance
becomes a straight line:
a two-dimensional line which hides
three-dimensional heart beats.

CHAPTER 8

H_v-MATRIX REPRESENTATIONS

8.1 THE PROBLEM

The problem of matrix representations of H_v-groups can be faced in two ways: First, by considering ordinary matrices over a field and defining a hyperproduct on matrices; however, one needs a standard definition of this hyperoperation. Second, by considering hypermatrices and the usual multiplication. In this book the second case is presented and it is studied mainly by using the fundamental relations.

H_v-matrix is called a matrix with entries elements of a H_v-semiring. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), which are of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner

$$(a_{ij})(b_{ij}) = \{ (c_{ij}) : c_{ij} \in C = \bigcup_{p\in P_n} p(a_{i1}b_{1j}, a_{i2}b_{2j}, \ldots, a_{in}b_{nj}) \} =$$

$$= a_{i1}b_{1j} \oplus a_{i2}b_{2j} \oplus \ldots \oplus a_{in}b_{nj} =: \oplus\sum a_{ik}b_{kj}$$

where $\oplus$ denotes the circle hyperoperation.

Remark that in the case of 2×2 H_v-matrices, the circle hyperoperation $\oplus$ is not actually used because it coincides with $+$.

Now the general problem of matrix representations of H_v-groups can be stated as follows:

8.1.1 PROBLEM

Consider a H_v-group $(H,\cdot)$. Find a H_v-semiring R, a set $M_R = \{(a_{ij}): a_{ij} \in R\}$ and a map $T: H \longrightarrow M_R: h \longmapsto T(h)$ such that

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \emptyset \quad \text{for all } h_1, h_2 \text{ in } H.$$

The map T is called **H_v-matrix representation**, and if the inclusion $T(h_1h_2) \subseteq T(h_1)T(h_2)$ for all h_1, h_2, is valid, then it is called **inclusion representation**. Finally, if the **good condition** is valid

$$T(h_1h_2) = T(h_1)T(h_2) = \{ T(h): h \in h_1h_2 \} \quad \text{for all } h_1, h_2 \in H$$

then, obviously, an induced representation T^* for the hypergroup algebra is obtained. A H_v-matrix representation is of **degree n**, if the set of H_v-matrices is of type n×n. It is called **faithful**, if it is one to one and the good condition is valid.

Obviously, the problem of representations can be simplified if the H_v-matrices are over a H_v-ring with 0 and 1 elements and even more if they are over such H_v-fields.

Finally, we shall mainly use COW H_v-rings and COW H_v-fields.

8.1.2 EXAMPLE

Let us denote the elements of $Z_2 \times Z_3$ by

$$\mu_1=(\bar{0},\bar{0}),\ \mu_2=(\bar{0},\bar{1}),\ \mu_3=(\bar{0},\bar{2}),\ \mu_4=(\bar{1},\bar{0}),\ \mu_5=(\bar{1},\bar{1}),\ \mu_6=(\bar{1},\bar{2}).$$

Then, consider the hyperoperation (*) which is the usual addition (+), except for the following cases

$$\mu_2{}^*\mu_2 = \mu_5{}^*\mu_5 = \{\mu_2,\mu_3\},\quad \mu_3{}^*\mu_6 = \mu_6{}^*\mu_3 = \{\mu_5,\mu_6\}$$

The hyperstructure $H=(Z_2\times Z_3,{}^*)$ is a H_b-group with $(Z_2\times Z_3,+)$ a b-group.

A faithful H_v-matrix representation of H can be given as follows:

Consider the minimal H_b-field which is non-degenerate, additive, with 0,1 scalar elements. This is, in fact, the WASS and weak distributive H_b-field $R = (Z_3,\oplus,\odot)$ where only the sum $\bar{2}\oplus\bar{2} = \{\bar{1},\bar{2}\}$ is not singleton and in the rest, the $\oplus$ and $\odot$ coincide with $(+),(\cdot)$, respectively, i.e. $(Z_3,+,\cdot)$ is a b-field of R (it is WASS, not associative since for example $\bar{1}+(\bar{2}+\bar{2})=\{\bar{1},\bar{2}\}$ but $(\bar{1}+\bar{2})+\bar{2}=\bar{2}$). Then, take the map $T:\ \mu_i \longmapsto M_i$, $i\in\{1,\ldots,6\}$ where the M_i's are the following 2x2 H_v-matrices over R

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad M_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\quad M_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\quad M_5 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\quad M_6 = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

Now one can easily check that T is a faithful representation of the H_b-group H of degree 2 over R. Note that

$$H/\beta^* = \{\ \{\mu_1,\mu_2,\mu_3\},\{\mu_4,\mu_5,\mu_6\}\ \}.\quad \text{So}\quad H/\beta^* \cong Z_2\ .$$

Examples for hypergroup matrix representation are the following:

8.1.3 EXAMPLES

In the set $I_n=\{0,1,\ldots,n\}\subset Z$ we define a commutative hyperoperation (*) as follows

$$\kappa*\lambda = \begin{cases} \max(\kappa,\lambda) & \text{if} \quad \min(\kappa,\lambda)=0 \\ \{0,1,\ldots,\ \min(\kappa,\lambda)-1,\ \max(\kappa,\lambda)\} & \text{if} \quad \min(\kappa,\lambda)>0 \end{cases}$$

This hyperoperation is associative because if we denote by x the one of κ,λ,μ between $\min(\kappa,\lambda,\mu)$ and $\max(\kappa,\lambda,\mu)$, then we have

$$(\kappa*\lambda)*\mu = \kappa*(\lambda*\mu) = \begin{cases} \max(\kappa,\lambda,\mu) & \text{if} \quad x=0 \\ \{0,1,\ldots,\ x-1,\ \max(\kappa,\lambda,\mu)\} & \text{if} \quad x>0 \end{cases}$$

Moreover, we observe that $\kappa*I_n=I_n*\kappa=I_n$ for all $\kappa\in I_n$, so $(I_n,*)$ is a hypergroup in the sense of F.Matry.

We also define a commutative operation ($\circ$) in I_n by the relation

$$\kappa\circ\lambda = \{\ \min(\kappa,\lambda)\ \}$$

which is, obviously, associative and one can see that it is strongly distributive with respect to (*). Thus $(I_n,*,\circ)$ is a hyperring. We consider the mapping from I_n into the set of 2x2 hypermatrices with entries from $(I_n,*,\circ)$:

$$\kappa \longmapsto \begin{bmatrix} n & \kappa \\ 0 & n \end{bmatrix} = T(\kappa)$$

Then we have

$$T(\kappa)T(\lambda) = \begin{bmatrix} n & \kappa \\ 0 & n \end{bmatrix}\begin{bmatrix} n & \lambda \\ 0 & n \end{bmatrix} = \begin{bmatrix} n & \lambda*\kappa \\ 0 & n \end{bmatrix} = \left\{ \begin{bmatrix} n & \rho \\ 0 & n \end{bmatrix} : \rho\in\kappa*\lambda \right\} =$$

$$= \{\ T(\rho) : \rho\in\kappa*\lambda\ \}$$

Hence, we have a representation of $(I_n,*)$ by 2x2 hypermatrices with entries from the hyperring $(I_n,*,\circ)$ and the good condition is satisfied.

Note that one can define $(*)$ and $(\circ)$, instead on the set I_n, in $I_{[m,n]}=\{m, m+1, \ldots, n\}$ for all $m,n\in Z$ with $n>m$.

Then $(I_{[m,n]}, *, \circ)$ is also a hyperring.

A special case is for $I_1=\{0,1\}$ where we have

$$0*0=0,\quad 0*1=1*0=1,\quad 1*1=\{0,1\} \quad \text{and} \quad 0\circ 0=0\circ 1=1\circ 0=0,\quad 1\circ 1=1.$$

Using this Boolean hyperring one can represent, apart of the hypergroup $(I_1, *)$, the ordinary group $\{a,b\}$ with two elements by setting

$$a \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Similarly, the mapping

$$a \longmapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b \longmapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

is a representation of the hypergroup $\{a,b\}$ with

$$aa = b, \qquad ab = ba = bb = \{a,b\}.$$

8.1.4 PROPERTIES

Let $(H, \cdot)$ be a H_v-group and T be a representation. Then

(a) T is a representation of $(H,*)$ for every $(*)$ such that $\cdot \leq *$.

(b) If T is inclusion, then it is also an inclusion representation for all $(H,*)$ such that $* \leq \cdot$.

Proof: Obvious.

8.1.5 REMARK (ONE-DIMENSIONAL CASE)

Consider all H_v-fields $(F,+,\cdot)$ obtained from the Constructions S_2, S_3, S_4 and S_5 . Then all H_v-groups which are embeddable (weak) in $(F,\cdot)$, have one-dimensional representations over F for obvious reasons.

For the one-dimensional representations of a H_v-group $(H,\cdot)$ we can consider the three types of **associated H_v-rings** of H defined in Theorem 3.2.2. Moreover, for hypergroups, consider the three types of associated hyperrings and complete associated semihyperrings defined in Remark 3.2.3. Using the identity mapping, it is clear that every H_v-group has one-dimensional representations over the associated H_v-rings and the complete associated H_v-semirings. However, in all the above types we have $\gamma^*(x)=H$ or H' for all $x\in H$ or H' although they are faithful.

8.2 A NECESSARY CONDITION

8.2.1 THEOREM

A necessary condition in order to have an inclusion representation T of the H_v-group $(H,\cdot)$ by nxn H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:

For every fundamental class $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in R$, $i,j\in\{1,\ldots,n\}$ such that

$$T(\beta^*(x)) \subset \{ A=(a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i,j\in\{1,\ldots,n\} \}$$

Proof

Let $x\beta^*y$ $x,y\in H$, and $T(x)=(x_{ij})$, $T(y)=(y_{ij})$ where $x_{ij},y_{ij} \in R$ for all $i,j\in\{1,\ldots,n\}$. Then, there exist elements

$z_1,\ldots,z_{r-1},h_{q_1},\ldots,h_{q_r} \in H$ and $Q_1,\ldots,Q_r$ finite sets of indices, such that setting $z_0=x$, $z_r=y$, there exist patterns $p_v \in P_{|Q_v|}$ for which

$$\{ z_{v-1}, z_v \} \subset \prod_{p_v,q_v\in Q_v} h_{q_v} , \quad v \in \{1,\ldots,r\} .$$

Therefore, for all $v\in\{1,\ldots,r\}$, we obtain for the hypermatrices

$$\{ T(z_{v-1}),\ T(z_v) \} \subset T\left(\prod_{p_v, q_v \in Q_v} h_{q_v} \right) \subset \prod_{p_v; q_v \in Q_v} T(h_{q_v}) .$$

In the ij entry of the H_v-matrix $\prod_{p_v, q_v \in Q_v} T(h_{q_v})$ there are sets which are the union of sets U_μ formed by sums of $n^{|Q_v|-1}$ products (n-circle hyperoperation) of Q_v elements each. Consequently, each of these sets U_μ, belongs to one γ^* fundamental class. Furtermore, there exists for every U_μ , another U_λ such that $U_\mu \cap U_\lambda \neq \emptyset$. Thus, since γ^* is transitive, it is obtained that there exists an element $a_{v,ij} \in R$ such that for the ij entries $z_{v-1,ij}$ and $z_{v,ij}$ of the H_v-matrices $T(z_{v-1})$ and $T(z_v)$, respectively, we have

$$\{ z_{v-1,ij} , z_{v,ij} \} \subset \gamma^*(a_{v,ij}) \quad \text{for all} \quad v \in \{1, \ldots, r\}.$$

So, again from the transitivity of γ^*, we have

$$\gamma^*(a_{1,ij}) = \ldots = \gamma^*(a_{v,ij}) =: \gamma^*(a_{ij}) .$$

Thus, we obtain, in particular,

$$\{ x_{ij}, y_{ij} \} \subset \gamma^*(a_{ij}) \quad \text{for all} \quad i,j \in \{1, \ldots, n\}. \qquad \blacksquare$$

8.2.2 REMARK

From the above proof it is obtained that if $A=(a_{ij})$, $B=(b_{ij})$ are H_v-matrices over $(R,+,\cdot)$, then every entry ij of the hyperproduct AB has elements of the same fundamental class γ^*. Thus $\gamma^*(\oplus\sum a_{ik} b_{kj})$ is an element of R/γ^* and we can denote this class by $\gamma^*(\sum a_{ik} b_{kj})$, since we have the same class for any pattern of parentheses used in the hypersum. Moreover, obviously,

$$\gamma^*((x_1 \oplus \ldots \oplus x_n) \oplus x_{n+1}) = \gamma^*(x_1 \oplus \ldots \oplus x_{n+1}) \quad \text{for all} \quad x_i \in R.$$

This remark is general for the product of H_v-matrices and it does not depend on the representations. Finally, for the same reason all H_v-matrices of (AB)C and A(BC) have in each entry ij elements of the same γ^* class.

According to this remark and using Theorem 8.2.1 the following necessary condition for H_v-matrix representations is immediate:

8.2.3 THEOREM

Every inclusion representation $T(a)=(a_{ij})$ of a H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+,\cdot)$ induces a homomorphic n×n representation T^* of the fundamental group H/β^* over the fundamental ring R/γ^* by setting

$$T^*(\beta^*(a)) = \left(\gamma^*(a_{ij}) \right) \quad \text{for all} \quad \beta^*(a) \in H/\beta^* ,$$

where the element $\gamma^*(a_{ij}) \in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. We call the T^*, **fundamental induced representation** of T, and we write fir.

8.2.4 DEFINITION

Let H be H_v-group and ω_H be the core of H. Two elements $x,y \in H$ are said to be **conjugate** if there exists an element z which has an inverse z^{-1}: $zz^{-1} \cap z^{-1}z \cap \omega_H \neq \emptyset$, such that $y \in z^{-1}(xz) \cap (z^{-1}x)z$.

8.2.5 THEOREM

Let T be an inclusion representation of H over R of degree n and let T^* be the fir of T. Denote I_n the unit matrix over R/γ^*, we have

(i) $T^*(u) = I_n$ for all $u \in \omega_H$

(ii) $T^*(z^{-1}) = (T^*(z))^{-1}$ for all inverse elements of z (even for partials)

(iii) $T^*(xy) = T^*(yx)$ $x,y \in H$

(iv) $T^*(z^{-1}(xz)) = T^*((z^{-1}x)z) = T^*(x) \quad x,y\in H$

Proof

(i) For each $x\in H$ there exists a partial identity u, i.e. $x\in xu$. Obviously $u\in\omega_H$. Thus $T(x) \in T(xu) \subset T(x)T(u)$ and, from Theorem 8.2.3, it is obtained $T^*(x) = T^*(x)T^*(u)$. Therefore, $T^*(u) = I_n$.

(ii) Let z^{-1} be an inverse element of z and let $u\in z^{-1}z$, $u\in\omega_H$. We obtain

$$T(u) \in T(z^{-1}z) \subset T(z^{-1})T(z)$$

and, going to the fir of T, we have

$$T^*(u) = T^*(z^{-1})T^*(z) = I_n \quad \text{so} \quad T^*(z^{-1}) = (T^*(z))^{-1}.$$

(iii) and (iv) The proofs are similar to (ii) taking first the T and then going to the fir of T.

The property (iv) says that T^* is a class function. ∎

A stricter condition of the one in Theorem 8.2.1 can be proved for hypergroups. From this point up to the end of this paragraph we suppose that we have hypergroups in the sense of Marty and hyperrings as well as inclusion homomorphisms.

8.2.6 DEFINITION

Let $(R,+,\cdot)$ be a hyperring. The **n-fundamental relation** $\underline{n}$ is defined in R as follows:

(i) $a\underline{n}a$ for all $a\in R$

(ii) $a\underline{n}b$, $a\neq b$, if there are $x_0,x_1,\ldots,x_k$ of R, where $x_0=a$, $x_k=b$, and $i_1,i_2,\ldots,i_k>1$ natural numbers such that

$$\{x_{s-1},x_s\} \subset \sum_{v=1}^{n^{i_s-1}} \left(\prod_{\mu=1}^{i_s} x_{s\mu v} \right), \quad s=1,\ldots,k \quad \text{and} \quad x_{s\mu v}\in R.$$

This is an equivalence relation and let $F_n(x)$ be the n-fundamental

class of x and F_n be the n-quotient. In F_n two hyperoperations are defined in the usual manner

$$F_n(x)+F_n(y) = \{ F_n(z) \mid z\in x'+y',\ \forall x' \underline{n} x,\ y' \underline{n} y \}$$

$$F_n(x)\cdot F_n(y) = \{ F_n(z) \mid z\in x'y',\ \forall x' \underline{n} x,\ y' \underline{n} y \}$$

for every $F_n(x),\ F_n(y) \in F_n$.

8.2.7 PROPOSITIONS

(i) If $\{u,v\}\subset xy$ for some $x,y\in R$, then $u \underline{n} v$.

(ii) $F_n(x)\cdot F_n(y) = \{ F_n(z) \}$ for all $z\in xy$.

(iii) The $\langle F_n, +, \cdot \rangle$ is an additive hyperring.

Proof

(i) From the reproduction axiom with respect to (+) one can choose $y_1,\ldots,y_n\in R$ such that $y\in y_1+\ldots+y_n$, then

$$\{u,v\} \subset xy \subset x(y_1+\ldots+y_n) \subset xy_1+\ldots+xy_n$$

therefore, from the definition it is obtained $u \underline{n} v$.

(ii) Let

$$F_n(x)\cdot F_n(y) = \{ F_n(z) \mid z\in x'y',\ \forall x' \underline{n} x,\ y' \underline{n} y \} \qquad (1)$$

Suppose $x' \underline{n} x$ and $y' \underline{n} y$ then $\exists x_0, x_1, \ldots, x_k \in R$ with $x_0 = x', x_k = x$ and $\exists y_0, y_1, \ldots, y_\lambda \in R$ with $y_0 = y', y_\lambda = y$ such that

$$\{x_{s-1}, x_s\} \subset \sum_{v=1}^{n^{i_s - 1}} \left(\prod_{\mu=1}^{i_s} x_{s\mu v} \right), \quad s=1,\ldots,k \text{ and } x_{s\mu v}\in R \qquad (2)$$

$$\{y_{t-1}, y_t\} \subset \sum_{v=1}^{n^{i_t - 1}} \left(\prod_{\mu=1}^{i_t} y_{t\mu v} \right), \quad t=1,\ldots,\lambda \text{ and } y_{t\mu v}\in R \qquad (3)$$

From the reproduction axiom, choose $u_1,\dots,u_n,v_1,\dots,v_n\in R$ such that $y'\in u_1+\dots+u_n$, $x\in v_1+\dots+v_n$. Using these, multiply all the members in (2) on the right side by y' and all members in (3) on the left side by x, so, for appropriate $\bar{x}$, $\bar{y}$, it is obtained

$$x_{s-1}y'\cup x_s y' \subset \sum_{v=1}^{n^{1_s}} \left(\prod_{\mu=1}^{1_s+1} \bar{x}_{s\mu v} \right), \quad s=1,\dots,k \tag{4}$$

$$xy_{t-1}\cup xy_t \subset \sum_{v=1}^{n^{1_t}} \left(\prod_{\mu=1}^{1_t+1} \bar{y}_{t\mu v} \right), \quad t=1,\dots,\lambda \tag{5}$$

From (4) and (5) take only one element of the sets $x_s y'$, xy_t for $s=1,\dots,k$, $t=2,\dots,l$ and observe that $x_k y'=xy_o$. Then it is obtained that all the elements of the set $x'y'=x_o y'$ are in n-fundamental relation with all the elements of the set $xy=xy_\lambda$. Therefore, the (1) has the form

$$F_n(x)\cdot F_n(y) = \{ F_n(z) \mid z\in xy \}$$

hence (ii) is proved.

(iii) Using the above properties, it remains to prove the inclusion distributivity. For $x,y,z\in R$ it is obtained

$$F_n(x)(F_n(y)+F_n(z)) = \bigcup_{\substack{w\in y'+z' \\ \forall y'\underline{n}y,\, z'\underline{n}z}} F_n(x)F_n(w) =$$

$$= \{ F(u) \mid u\in xw,\ w\in y'+z',\ \forall y'\underline{n}y,\ z'\underline{n}z \} \subset$$

$$\subset \{ F(u) \mid u\in xy'+xz',\ \forall y'\underline{n}y,\ z'\underline{n}z \} \subset$$

$$\subset \{ F(u) \mid u\in t'+s',\ \forall t'\underline{n}t,\ s'\underline{n}s \text{ where } t\in xy,\ s\in xz \} =$$

$$= F_n(t)+F_n(s) \text{ (where } t\in xy,\ s\in xz) = F_n(x)F_n(y)+F_n(x)F_n(z). \quad \blacksquare$$

8.2.8 PROPOSITIONS

Let the hyperring R endowed with a zero element (0) and a unit element(1), then

(i) If $\{u,v\} \subset x+y$ for some $x,y \in R$, then $u \underline{n} v$ for $n>1$.

(ii) $F_n(x)+F_n(y) = \{F_n(y)\}$ for all $z \in x+y$.

(iii) $\beta^*_+(x) \subset F_n(x)$, $n>1$.

Proof

(i) $$\{u,v\} \subset x+y+\underbrace{0+\ldots+0}_{n-2 \text{ zero's}} \subset (x+y+0+\ldots+0)\cdot 1 \subset x\cdot 1+y\cdot 1+0\cdot 1+\ldots+0\cdot 1$$

therefore, $u \underline{n} v$.

Note that for the 2-fundamental relation it is enough to have the unit element.

(ii) Let $F_n(x)+F_n(y) = \{ F_n(z) \mid z \in x'+y',\ \forall x' \underline{n} x,\ y' \underline{n} y \}$

Multiply by 1 all the second members in the relations (2) and (3) of the Proposition 8.2.7(ii), and add in (2) on the right side, in both members, respectively the inclusions

$$\{y'\} \subset y'\cdot 1^{i_s}+\underbrace{0\cdot 1^{i_s}+\ldots+0\cdot 1^{i_s}}_{n^{i_s}-n^{i_s-1}-1 \text{ times}} \quad \text{where } s=1,\ldots,k$$

Furthermore, add in (3) on the left side, in both members, respectively, the inclusions

$$\{x\} \subset x\cdot 1^{i_t}+\underbrace{0\cdot 1^{i_t}+\ldots+0\cdot 1^{i_t}}_{n^{i_t}-n^{i_t-1}-1 \text{ times}} \quad \text{where } t=1,\ldots,l$$

Then, in a similar way as in the above proposition, it is obtained that the elements of the set $x'+y'$ are in n-fundamental relation with the elements of $x+y$. So by (i) the required result is obtained.

(iii) From $\{x,y\} \subset z_1+\ldots+z_k$ it is obtained that

$$\{x,y\} \subset z_1\cdot 1+\ldots+z_k\cdot 1+\underbrace{0\cdot 1+\ldots+0\cdot 1}_{n-k \text{ times}} \qquad \text{if } k\le n$$

$$\{x,y\} \subset z_1\cdot 1\cdot 1+\ldots+z_k\cdot 1\cdot 1+\underbrace{0\cdot 1\cdot 1+\ldots+0\cdot 1\cdot 1}_{n^2-k \text{ times}} \qquad \text{if } n<k\le n^2$$

and so on. That means that if $x\beta^* y$ then $x\eta y$. ∎

Remark that from this Proposition it is obtained that if R has a zero and a unit then $\langle F_n,+,\cdot\rangle$, $n>1$, is a ring. Moreover if this zero and unit is absolute, then the definition of the n-fundamental relation is simplifyied as follows: for $s=1,\ldots,k$

$\{x_{s-1},x_s\}$ is a subset of a sum of products of elements of R.

Finally in this case $F_n(x) \subset F_{n+1}(x)$ for every n.

8.2.9 THEOREM

A necessary condition in order to have a representation T of the hypergroup H by nxn hypermatrices over the hyperring R is the following:

For any class $\beta^*(x)$, $x\in H$, there must exist elements $x_{ij}\in R$ where $i,j=1,\ldots,n$ such that

$$T(\beta^*(x)) \subset \{ (x'_{ij}) \mid x'_{ij} \in F_n(x_{ij}),\ i,j=1,\ldots,n \}$$

where (x'_{ij}) denotes the nxn hypermatrix with ij entry x'_{ij} .

Proof

Let $x,y\in H$ such that $x\beta^* y$ and $T(x)=(x_{ij})$, $T(y)=(y_{ij})$ where $x_{ij},y_{ij}\in R$ for all $i,j=1,\ldots,n$. Then there are $x_1,\ldots,x_{k-1};h_{i_1},\ldots,h_{i_k}\in H$ and $I_1,\ldots,I_k$ finite sets of indices such that

$$\{x,x_1\} \subset \prod_{i_1\in I_1} h_{i_1}\,,\ \ldots\ ,\ \{x_{k-1},y\} \subset \prod_{i_k\in I_k} h_{i_k}$$

Therefore, denoting by $\cdot\prod$ the product of hypermetrices, we have

$$\{T(x),T(x_1)\} \subset T\Big(\prod_{i_1\in I_1} h_{i_1}\Big) \subset \cdot\prod_{i_1\in I_1} T(h_{i_1}) ,$$

. . . . . .

$$\{T(x_{k-1}),T(y)\} \subset T\Big(\prod_{i_k\in I_k} h_{i_k}\Big) \subset \cdot\prod_{i_k\in I_k} T(h_{i_k}) .$$

Let's denote $T(h_{i_\lambda}) = (h_{i_\lambda,ij})$ where $h_{i_\lambda,ij}\in R$ and $i,j=1,\dots,n$; $\lambda=1,\dots,k$, then in the ij position of the products

$$\cdot\prod_{i_\lambda\in I_\lambda} T(h_{i_\lambda}) = \cdot\prod_{i_\lambda\in I_\lambda} (h_{i_\lambda,ij}) , \quad \lambda=1,\dots,k$$

there are sets of the form

$$\sum_{v=1}^{n} \Big(\prod_{\mu\in I_\lambda} z_{\mu_v,ij}\Big)^{|I_\lambda|-1} .$$

So for the transitive closure we deduce that in the ij position of all hypermatrices

$$T(x),\ T(x_1),\ \dots,\ T(x_{k-1}),\ T(y)$$

we have elements of R which are in n-fundamental relation, therefore $x_{ij}\ \underline{n}\ y_{ij}$. ∎

We do not know yet if we can tranfer this theorem to H_v-groups and H_v-rings and which modifications are needed. For the one dimensional representations, one can use the associated and complete associated semihyperrings.

Another direction to face the above problem, is to find special semihyperrings or hyperrings.

8.3 THEOREMS

8.3.1 THEOREM

Let $(G,\cdot)$ be a group and P be a normal subgroup of G. Suppose G/P has finite order n, and fix a representative $\bar{g}$ of each class gP. Then consider the P-hypergroup $\langle G,P^*\rangle$, defined by setting $xP^*y=xPy$ for all $x,y\in G$. Consider the H_v-ring $\mathbb{Z}_2^2$, where we replace by A an element different from $0,1$, defined in the Example 1.5.8. For this, we take the set $A=\{a_i \mid i\in I\}$ with $\mathrm{card}A = \mathrm{card}P$, and a 1-1 mapping $a_i \longmapsto p_i$, $i\in I$. Let T_μ be the monomial representation of the group G/P. Then the mapping

$$T : g \longrightarrow T(g) = \begin{pmatrix} T_\mu(gP) & 0 \\ 0 & a_i \end{pmatrix} \quad \text{where} \quad g = \bar{g}p_i \ , \ p_i\in P,$$

is a faithful $(n+1)\times(n+1)$ H_v-matrix representation over R.

Proof

For all $g_1,g_2\in G$ we have

$$T(g_1P^*g_2) = T(g_1Pg_2) = \{\, T(g_1p_sg_2) \mid p_s \in P \,\} =$$

$$= \left\{ \begin{pmatrix} T_\mu(g_1p_sg_2P) & 0 \\ 0 & a_t \end{pmatrix} \mid p_s \in P \quad \text{where} \quad g_1p_sg_2 = \overline{g_1p_sg_2}\ p_t \right\}.$$

But since p_s runs all P, we obtain

$$T(g_1P^*g_2) = \left\{ \begin{pmatrix} T_\mu(g_1g_2P) & 0 \\ 0 & a_t \end{pmatrix} \mid t \in I \right\} = \begin{pmatrix} T_\mu(g_1g_2P) & 0 \\ 0 & A \end{pmatrix}.$$

On the other hand we have, for $g_1 = \bar{g}_1p_1$, $g_2 = \bar{g}_2p_j$,

$$T(g_1)T(g_2) = \begin{bmatrix} T_\mu(g_1P) & 0 \\ 0 & a_i \end{bmatrix} \begin{bmatrix} T_\mu(g_2P) & 0 \\ 0 & a_j \end{bmatrix} = \begin{bmatrix} T_\mu(g_1g_2P) & 0 \\ 0 & A \end{bmatrix}.$$

Thus $T(g_1P^*g_2) = T(g_1)T(g_2)$ for all $g_1, g_2 \in G$. ■

We will generalize the above theorem but first, we define a special mapping f_M as follows:

Let M be a monomial nxn matrix and $A=\{a_i \mid i\in I\}$ be a set. We consider the mapping f_M which for each element $(a_1,\ldots,a_n) \in A^n$ corresponds the H_v-matrix $M(a_1,\ldots,a_n)$, which has for all $i\in\{1,\ldots,n\}$, in the i row, the element a_i instead of the unit.

8.3.2 THEOREM

Let $(G,\cdot)$ be a group with $|G| = nt^n$, $n,t\in N^*$. Suppose there exists a normal subgroup P of G with $|P|=t^n$, then consider the P-hypergroup $\langle G,P^*\rangle$ defined by setting $g_1P^*g_2 = g_1Pg_2$ for all $g_1, g_2\in G$. The $\langle G,P^*\rangle$ has a faithful representation T over the H_v-ring R obtained from Z_2^2 where one non-zero and non-unit element is replaced by a set A with $|A|=t$. The T is given as follows:

Take the monomial representation of G/P and fix a representative g_o of each class, so $gP=g_oP \longmapsto M_{g_o}$. Since $|P| = |A|^n$, take a 1-1 mapping from P to A^n and denote $p_x \longmapsto (x_1,\ldots,x_n)$, $x_i\in A$. Afterwards, for each g of gG corresponds the element $(z_1,\ldots,z_n)\in A^n$, where $p_z = gg_o^{-1}\in P$.

Thus $f_M(g) = M_{g_o}(z_1,\ldots,z_n)$, where $p_z=gg_o^{-1}$ for all $g\in G$.

Proof

Let $g,g'\in G$, then $T(g) = M_{g_o}(z_1,\ldots,z_n)$ where $p_z = gg_o^{-1}$ and

$T(g') = M_{g_o'}(z_1', \ldots, z_n')$ where $p_{z'} = g'(g_o')^{-1}$. Thus

$$T(gP^*g') = T(gPg') = \{ T(gpg') \mid p \in P \} =$$

$$= \{ M_{(gpg')_o}(u_1, \ldots, u_n) \mid p \in P \text{ where } p_u = (gpg')(gpg')_o^{-1} \} .$$

But $(gpg')_o = (gg')_o$, and since $p \in P$, p_u runs all the elements of P and, consequently, we obtain:

$$T(gP^*g') = \{ M_{(gg')_o}(u_1, \ldots, u_n) \mid p \in P \text{ where } p_u = (gpg')(gg')_o^{-1} \} =$$

$$= \{ M_{(gg')_o}(u_1, \ldots, u_n) \mid (u_1, \ldots, u_n) \in A^n \}$$

On the other hand, we have

$$T(g)T(g') = M_{g_o}(z_1, \ldots, z_n) \cdot M_{g_o'}(z_1', \ldots, z_n') = M_{g_o g_o'}(A^n),$$

and, since $M_{(gg')_o} = M_{g_o g_o'}$, the theorem was proved. ■

8.3.3 COROLLARY

Let $(G, \cdot)$ be a group such that $G = G_1 \oplus \ldots \oplus G_m$ where

$$|G_i| = n_i t_i^{n_i}, \quad i \in \{1, \ldots, m\}, \quad \text{so} \quad |G| = n_1 t_1^{n_1} \ldots n_m t_m^{n_m}.$$

Suppose there exists normal subgroup P_i of G_i, with $|P_i| = t_i^{n_i}$, for all $i \in \{1, \ldots, m\}$. Consider the P-hypergroup $\langle G, P^* \rangle$, with

$P = P_1 \oplus \ldots \oplus P_m$, where $(g_1, \ldots, g_m)P^*(g_1', \ldots, g_m') = (g_1 P_1 g_1', \ldots, g_m P_m g_m')$,

for all $(g_1, \ldots, g_m), (g_1', \ldots, g_m') \in G$. Then, the $\langle G, P^* \rangle$ has a faithful representation T over the ring R, obtained from Z_2^s, with

$2s \geq m$, where the m of the non-zero and non-unit elements have been replaced by the sets A_i with $|A_i| = t_i$, $i \in \{1,\dots,m\}$.

The construction of T is similar to the above theorem.

The proof can be carried out on the direct sum of the representations of each component, from the fact that in the H_v-ring R there exist 0 and 1 which are mutually scalars.

8.3.4 REMARK

In the special case of the above Corollary for which $G = G_1 \oplus G_2$ and $P_1 = \{e\}$, $n_2 = 1$, we have $|G| = n_1 \cdot 1^{n_1} \cdot 1 \cdot t_2^1 = n_1 t_2$. Thus the Theorem 8.3.1 is obtained again.

8.4 REPRESENTATIONS BY BOOLEAN H_v-MATRICES

Since every generalized permutation f on $X = \{ x_1,\dots,x_n \}$ is a relation on X, there exists the incidence matrix $T(f) = (f_{ij})$ of f. This is defined by setting

$$f_{ij} = \begin{cases} 1 & \text{if } x_j \in f(x_i) \\ 0 & \text{otherwise} \end{cases}$$

In this representation suppose that the Boolean or switching algebra on $\{0,1\}$ is the semiring defined by the relations

$$0+0=0,\ 0+1=1+0=1+1=1 \quad \text{and} \quad 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0,\ 1 \cdot 1 = 1 ,$$

i.e. we consider the join and the meet operations.
Since $x_j \in f(x_i) \iff x_i \in \bar{f}(x_j)$ remark that the corresponding matrix $(\bar{f}_{ij})$ to the converse permutation $\bar{f}$ is transpose to the (f_{ij}). If $f, g \in M_X$, then the multiplication is defined by setting

$f \cdot g = g \circ f$, i.e. $(f \cdot g)(x) = \bigcup_{y \in g(x)} f(y)$.

8.4.1 THEOREM

The map T is a faithful representation.

Proof

Let $f,g \in M_X$, then $T(f\cdot g) = T(g\circ f) = ((g\circ f)_{ij})$ where

$$(g\circ f)_{ij} = \begin{cases} 1 & \text{if } x_j \in (g\circ f)(x_i) \\ 0 & \text{otherwise} \end{cases}$$

and $T(f)\cdot T(g) = (f_{ij})(g_{ij}) = (c_{ij})$ where $c_{ij} = \sum_k f_{ik}\cdot g_{kj}$.

So it is enough to prove that $(g\circ f)_{ij} = c_{ij}$, $\forall i,j=1,\ldots,n$.

Let us suppose that $(g\circ f)_{ij}=1$, then $x_j \in (g\circ f)(x_i) \Rightarrow x_j \in g(f(x_i))$,

therefore, there exists a x_s such that $x_j \in g(x_s)$, $x_s \in f(x_i)$;

so $g_{sj}=1$, $f_{is}=1$ and $f_{is}\cdot g_{sj}=1$.

Consequently, $c_{ij} = \sum_k f_{ik}\cdot g_{kj} = 1$.

On the other hand, let us suppose that $c_{ij}=1$. Then there exists a s such that $f_{is}\cdot g_{sj} = 1$ or $f_{is} = g_{sj} = 1$, so $x_s \in f(x_i)$ and $x_j \in g(x_s)$.

So $x_j \in g(x_s) \subset g(f(x_i))$ or $x_j \in g(f(x_i))$, i.e. $(g\circ f)_{ij}=1$.

Therefore, T is an homomorphism. Moreover, T is 1-1, so T is a faithful representation. ∎

In the set of all nxn Boolean (incidence) matrices the **inclusion** "≤" is defined on $A = (a_{ij})$, $B = (b_{ij})$ by setting $A \le B$ iff $a_{ij} \le b_{ij}$ for all $i,j=1,\ldots,n$. It is immediate that T preserves the inclusion. The **universal switching matrix**, i.e. the **maximal**, is the one with all entries equal to 1 and the **zero switching matrix** is the one with all entries 0.

8.4.2 DEFINITION

An nxn incidence matrix will be called **reproductive incidence matrix** if in every row and column there exists at least one 1. We will denote by S_R the set of all reproductive incidence matrices.

8.4.3 THEOREM

The map $T : M_X \longrightarrow S_R : f \longmapsto (f_{ij})$, where (f_{ij}) is the incidence matrix of f, is an isomorphism.

Proof

For every i we have $f(x_i) \neq \emptyset$ so in every row there exists at least one 1. Furthermore, $X = f(X)$ so for every j there exists an s such that $x_j \in f(x_s)$; that means that in every column appears at least one 1. So $(f_{ij}) \in S_R$.

Now let $(a_{ij}) \in S_R$, then we can define the map $\alpha: X \longrightarrow \mathcal{P}(X)$ by setting $\alpha(x_i) = \{x_j: a_{ij} = 1\}$. This map is a generalized permutation since $\alpha(X) = \{ x_j: a_{ij} = 1, i=1,\ldots,n \} = X$, because for every j in the j column of (a_{ij}) there exists at least one 1, so $x_j \in \alpha(X)$. Therefore, T is a bijection.

Using Theorem 8.4.1 we obtain that T is an isomorphism. ∎

8.4.4 REMARKS

(i) Any minimal permutation corresponds to a minimal incidence matrix which is defined as follows: A Boolean matrix $A \in S_R$ will be called **minimal** if there is no matrix B of S_R such that $B<A$. We denote by $\underline{S}_R$ the set of all minimal matrices.

(ii) Let $f = f(x_1,\ldots,x_k;y_1,\ldots,y_k)$ be a switching function on 2k variables. Using a generalization of the Karnaugh map method we correspond to f a $2^k \times 2^k$ switching matrix as follows:

We put every minimal polynomial, i.e. every expression of the form

$\hat{x}_1 \cap \ldots \cap \hat{x}_k \cap \hat{y}_1 \cap \ldots \cap \hat{y}_k$ with $\hat{x}_i \in \{ x_i, \bar{x}_i \}$, $\hat{y}_j \in \{ y_j, \bar{y}_j \}$, $\forall i,j=1,\ldots,k$

on one cell(entry) of a $2^k \times 2^k$ matrix. The correspondence is such that to every expression $\hat{x}_1 \cap \ldots \cap \hat{x}_k$ corresponds a row of the matrix and to every expression $\hat{y}_1 \cap \ldots \cap \hat{y}_k$ corresponds a column of the matrix. The above method is very useful to simplify forms of the original switching circuit. The switching matrix of the f has entry $f(\hat{x}_1, \ldots, \hat{x}_k; \hat{y}_1, \ldots, \hat{y}_k)$ at the cell corresponding to the minimal polynomial $\hat{x}_1 \cap \ldots \cap \hat{x}_k \cap \hat{y}_1 \cap \ldots \cap \hat{y}_k$. That means that f is 1 or 0 if the minimal polynomial $\hat{x}_1 \cap \ldots \cap \hat{x}_k \cap \hat{y}_1 \cap \ldots \cap \hat{y}_k$ appears or not in a complete analysis of f into the minimal polynomials.

A switching function f corresponds to a reproductive matrix, so to a generalized permutation, if the following condition is valid: For any constants $s_1, \ldots, s_k; t_1, \ldots, t_k$ the switching functions

$$f(s_1, \ldots, s_k; y_1, \ldots, y_k) \quad \text{and} \quad f(x_1, \ldots, x_k; t_1, \ldots, t_k)$$

are not zero functions.

We notice that the above interpretation of the generalized permutations by Boolean matrices depends on the choice of those two sets of variables $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$.

Remark that the universal switching matrix corresponds to the universal permutation matrix which is reproductive. The zero matrix is not reproductive.

From the isomorphism $M_X \cong S_R$ which is obtained in Theorem 8.4.3 we have analogous theorems to the ones in Chapters 6 and 7. However, some analogous definitions are needed. For example one can see that the subset of S_R which corresponds to the generalized permutations of the set M_θ has matrices with the following form: There exists an appropriate permutation on the rows as well as a permutation on the columns such that the obtaining matrix can be divided in block which have everywhere 0 or everywhere 1.

Πάλι οι μαύρες ορδές
των γραμμάτων
γέμισαν προσταχές
τις επερχόμενες γενιές.

Again black gangs
of letters
stuffed future generations
with commands.

CHAPTER 9

REDUCIBILITY AND CHARACTERS

9.1 TRACES

The character is a very important function in the theory of representations because it characterizes the representation. Thus it is natural to define this for H_v-matrix representations as well. But first we need some definitions. In the following M_n denotes the set of all $n \times n$ H_v-matrices over a given H_v-ring.

9.1.1 DEFINITIONS

Let $(R,+,\cdot)$ be H_v-ring endowed with a zero element 0, and the set of unit elements $E_R = \{ e \in R : re \in (er) \cap (re) \text{ for all } r \in R \}$ is not empty.

A H_v-matrix $I_n \in M_n$ is called **unit H_v-matrix** if it is of the form

$$I_n = \mathrm{diag}(e_1,\ldots,e_n) \text{ where } e_i \in E_R \text{ for all } i \in \{1,\ldots,n\}.$$

So one has $(AI_n \cap I_n A) \ni A$ for all $A \in M_n$. Remark that the above relation can be also valid for non-diagonal H_v-matrices, but the set of identities becomes greater. An $A \in M_n$ will be called **invertible** if there exists a H_v-matrix $A^{-1} \in M_n$, called **inverse** of A, such that $(A^{-1}A \cap AA^{-1}) \ni I_n$ where I_n is a unit matrix.

These definitions may give enormous number of identities and inverses, however, more we are interested in H_v-rings endowed with unique 0 and 1. So we prefer they are both scalars or both singles where, from Theorem 1.3.8, they are unique. In the first case, where they are both scalars, we have $AI_n = I_n A = A$ for all $A \in M_n$ but still for the inverses $(A^{-1}A \cap AA^{-1}) \ni I_n$. In the second case, where R is H_v-single ring, $(AI_n \cap I_n A) \ni A$ but for all invertibles $A \in M_n$ it is obtained that $I_n = A^{-1}A = AA^{-1}$, i.e. the inverse is unique. Indeed, in each entry of $A^{-1}A$ and AA^{-1} it is contained 0 or 1, so from definition this entry must be singleton, in fact must be 0 or 1 respectively.

In the following we restrict ourselves mainly to H_v-single rings.

A H_v-matrix $A \in M_n$ is said to be **quasi-diagonal** if it is of the block form:

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_k)$$

where $A_i \in M_{n_i}$, $i=1,\ldots,k$, $n_i \in N^*$ and $n_1 + \ldots + n_k = n$

So A is the direct sum of the H_v-matrices A_i and we write

$$A = A_1 \oplus \ldots \oplus A_k .$$

$A \in M_n$ will be called **reducible** if there is an invertible H_v-matrix B such that $(B^{-1}A)B$ or $B^{-1}(AB)$ contains a proper quasi-diagonal $A' \in M_n$, i.e. A is similar to a quasi-diagonal H_v-matrix. If A is not reducible, then it is called **irreducible**.

9.1.2 DEFINITION

Let $A=(a_{ij})$ be H_v-matrix over the COW H_v-ring $(R,+,\cdot)$, then we can define the following traces:

(a) $Tr : M_n \longrightarrow \mathcal{P}(R) : TrA = \overset{n}{\underset{i=1}{\oplus\sum}} a_{ii}$, where $\oplus$ is the n-circle hyperoperation

(b) $tr_x : M_n \longrightarrow R : tr_xA \in TrA$, $x\in R$

(c) For H_b-rings, $tr_b : M_n \longrightarrow R : tr_bA = \sum b_i$,

where $\sum b_i$ is the basic element of the sum $\oplus\sum a_{ii}$ which is an element of the b-ring and n>1 (basic trace).

(d) $tr_\varphi : M_n \longrightarrow R/\gamma^* : tr_\varphi A = \gamma^*(\sum a_{ii}) = tr\varphi(A) \in R/\gamma^*$

where φ is the fundamental map (fundamental trace).

9.1.3 THEOREM

(i_a) $tr_\varphi(I_n) = n$ for all nxn unit H_v-matrices on R.

(i_b) $tr_\varphi(AB) = tr_\varphi(BA)$ for all $A,B \in M_n$.

(i_c) $tr_\varphi(B^{-1}(AB)) = tr_\varphi((B^{-1}A)B) = tr_\varphi A$ for all $A,B \in M_n$.

(ii_a) $tr_b(I) = n$ for the unique unit matrix of the b-ring.

(ii_b) $tr_b(AB) = tr_b(BA)$ for all $A,B \in M_n$.

(ii_c) $tr_b(B^{-1}(AB)) = tr_b((B^{-1}A)B) = tr_bA$ for all $A\in M_n$ and for invertible, with respect to the b-ring, $B\in M_n$.

Proof

(i_a) Every $e\in E_R$ is multiplicative partial identity so $\varphi(e)=1\in R/\gamma^*$, thus $\varphi(I_n)=I$, where I is the unit matrix over R/γ^*. Consequently, $tr_\varphi(I_n) = n$.

(i_b) Since R is COW, R/γ^* is commutative, so for all $A,B \in M_n$

$$tr_\varphi(AB) = \gamma^*\Big(\oplus\sum_i(\oplus\sum_k a_{ik}b_{ki})\Big) = \gamma^*\Big(\sum_i\sum_k a_{ik}b_{ki}\Big) =$$

$$= \gamma^*\Big(\sum_i\sum_k b_{ki}a_{ik}\Big) = tr_\varphi(BA).$$

(i_c) $tr_\varphi(B^{-1}(AB)) = tr\varphi(B^{-1}(AB)) = tr\varphi((AB)B^{-1}) = tr\varphi(A(BB^{-1})) =$

$$= tr(\varphi(A)\cdot\varphi(BB^{-1}))$$

But BB^{-1} contains a unit matrix of R, so $\varphi(BB^{-1}) = I$. Thus

$$tr_\varphi(B^{-1}(AB)) = tr_\varphi(A)$$

(ii_a), (ii_b) and (ii_c) are obvious since they become properties of the usual matrices over rings, i.e. the b-rings. ∎

Remark that for Tr and tr_x the above properties are not necessarilly valid. However, for H_b-matrices we obviously have

$$Tr(AB) \cap Tr(BA) \neq \emptyset \quad \text{for all} \quad A,B \in M_n .$$

9.1.4 DEFINITION

Let T be a representation of a H_v-group H by H_v-matrices over R, we shall call **fundamental character** of T the mapping

$$X_T : H \longrightarrow R/\gamma^* : x \longmapsto X_T(x) = tr_\varphi(T(x)) = trT^*(x)$$

If R is H_b-ring then we shall call **basic character** of T the mapping

$$X_{Tb} : H \longrightarrow R : x \longmapsto X_{Tb}(x) = tr_b T(x)$$

9.2 REPRESENTATIONS

9.2.1 DEFINITIONS

Let T be a H_v-matrix representation, of degree n, of H over R. Let's denote by

$$\mathbf{diag}(M_{n_1},\ldots,M_{n_k}),\quad \text{where } n_i\in\mathbb{N}^* \text{ and } n_1+\ldots+n_k = n,$$

the set of all quasi-diagonal H_v-matrices over R, $\mathrm{diag}(A_1,\ldots,A_k)$ where $A_{n_i}\in M_{n_i}$, $\forall i=1,\ldots,k$. If every matrix $T(x)$, $x\in H$ has a similar matrix which belongs to a fixed set

$$\mathbf{diag}(M_{n_1},\ldots,M_{n_k}),$$

then T is called **reducible**. If T is not reducible, then it is called **irreducible**. If T is reducible then for every $x\in H$ we can write

$$T(x) = \mathrm{diag}(T_1(x),\ldots,T_k(x)),\quad \text{where } T_i(x)\in M_{n_i},\ i=1,\ldots,k.$$

The components T_i, $i=1,\ldots,k$ are also representations of H over R. Indeed, it is clear that for all $x,y\in H$, we have

$$T(xy) = \mathrm{diag}(T_1(xy),\ldots,T_k(xy)) =$$

$$= \mathrm{diag}(T_1(x),\ldots,T_k(x))\cdot\mathrm{diag}(T_1(y),\ldots,T_k(y)) = T(x)T(y)$$

We write $T = T_1\oplus\ldots\oplus T_k$

and T is called **direct sum** of the representations $T_1,\ldots,T_k$.

In a direct sum it is immediate that

$$tr_\varphi T(x) = tr_\varphi T_1(x) + \ldots + tr_\varphi T_k(x),$$

and for H_b-rings

$$tr_b T(x) = tr_b T_1(x) + \ldots + tr_b T_k(x) \quad \text{for all } x \in H.$$

Therefore, the characters X_T and X_{Tb} can be written as

$$X_T = X_{T_1} + \ldots + X_{T_k} \quad \text{and} \quad X_{Tb} = X_{T_1 b} + \ldots + X_{T_k b}.$$

A character is called **reducible** (resp. **irreducible**) if it corresponds to reducible (resp. irreducible) representation.

9.2.2 THEOREM

Let T be any inclusion H_v-matrix representation of H over R, of degree n. Then there exists a unique group character X_T° such that $X_T = X_{T_\varphi} \circ \varphi$, of the fundamental group over the fundamental ring.

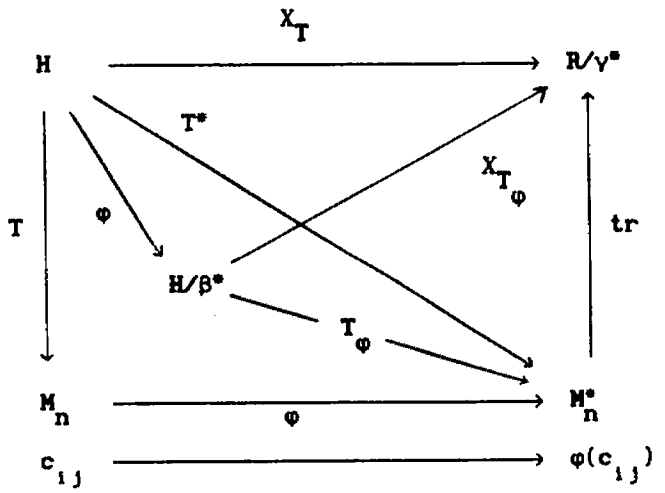
Proof

From the definition $X_T = tr \circ \varphi \circ T$, where φ is the natural mapping corresponding to each H_v-matrix (c_{ij}) the matrix over R/γ^*, which has the element $\varphi(c_{ij})$ in the ij entry. Moreover, the fir of T can be written as $T^* = \varphi \circ T$ and by Theorem 8.2.5(iv) is a class function. So considering the projection $\varphi: H \longrightarrow H/\beta^*$, there exists a representation $T_\varphi: H/\beta^* \longrightarrow M_n^*$, where M_n^* denotes the set of all nxn matrices over R/γ^*, such that $T^* = T_\varphi \circ \varphi$.

Thus the character of T_φ is defined by $X_{T_\varphi} = tr \circ T_\varphi$.

Now it is immediate that $X_T = X_{T_\varphi} \circ \varphi$.

The whole situation is, therefore, described by the following commutative diagram



This theorem states that for each X_T there exists a unique character which obeys the known theory of matrix representations of groups. ∎

Let $(H,\cdot)$ be H_b-group with b-group $(H,\circ)$, and let $(R,+,\cdot)$ be H_b-ring with b-ring $(R,\blacksquare,\circ)$. Then there exists

$f_H \in \mathrm{Aut}H$, $f_R \in \mathrm{Aut}R$ such that $x \circ y \in f_H(xy)$ for all $x,y \in H$,

and

$a \blacksquare b \in f_R(a+b)$, $a \circ b \in f_R(ab)$ for all $a,b \in R$.

Thus there are restriction mappings r_H, r_R such that

$$r_H : (H,\cdot) \longrightarrow (H,\circ) \quad \text{and} \quad r_R : (R,+,\cdot) \longrightarrow (R,\blacksquare,\circ)$$

for which

$$r_H \circ f_H(xy) = x \circ y \quad \text{for all } x,y \in H, \quad \text{and}$$

$$r_R \circ f_R(a+b) = a \blacksquare b, \quad r_R \circ f_R(ab) = a \circ b \quad \text{for all } a,b \in R.$$

Now we have the following:

9.2.3 THEOREM

For any H_v-matrix representation T the character X_{Tb} has a 1-1 correspondence with the group character of the b-group to the b-ring corresponding to the induced representation of T on the b-group.

Proof

Following the above remarks we have

$$X_{Tb} = tr \circ r_R \circ f_R \circ T$$

where $r_R \circ f_R : (c_{ij}) \longmapsto (r_R \circ f_R(c_{ij}))$.

Consider the 1-1 (forgetful) mapping

$$r_H \circ f_H : (H,\cdot)\times(H,\cdot) \longrightarrow (H,\odot)\times(H,\odot) : r_H \circ f_H(xy) = x \odot y.$$

Then there exists an induced representation T_i of T, from the b-group $(H,\odot)$ to the nxn matrices over $(R,\boxplus,\circ)$ such that

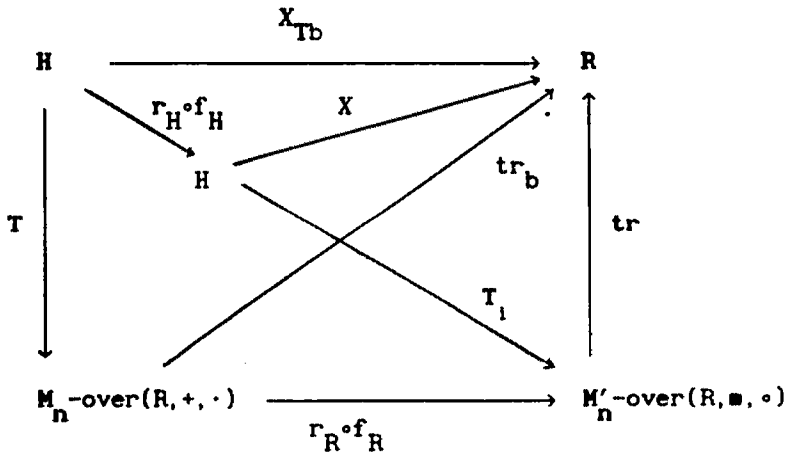
$$T_i = r_R \circ f_R \circ T \circ (r_H \circ f_H)^{-1}.$$

So denoting by X the character of T_i we have

$$X = tr \circ T_i = tr \circ r_R \circ f_R \circ T \circ (r_H \circ f_H)^{-1} = X_{Tb} \circ (r_H \circ f_H)^{-1}$$

but $(r_H \circ f_H)^{-1}$ is a 1-1 map, and therefore the theorem was proved.

Hence, this situation is described by the following commutative diagram:


X_{Tb}
H
R
$r_H \circ f_H$
X
H
tr_b
T
tr
T_1
M_n-over(R, +, ·)
M'_n-over(R, ■, ∘)
$r_R \circ f_R$

Κυκλοφόρησε
μια δεκάτομη μαγειρική
για να μας στερήσει
το γνώριμο φαγητό.

A ten-volume cook book
was just published, in order to
deprive us of
our favorite dishes.

APPENDIX 1

CLASSICAL HYPERSTRUCTURES

A1 CLASSICAL HYPERSTRUCTURES

With the term classical hyperstructures we mean here all the well known classical type hyperstructures based on the hypergroup in the sense of Marty. One can give the title "Krasnerian type hyperstructures" The reason to give this title, is that some of the basic hyperstructures were introduced and studied during Krasner's research activity, five decades, on this field. In this appendix we give only some of the basic definitions on the subject since a well and wide exposition is given in Corsini's book: Prolegomena of Hypergroup Theory.

In the following we assume that all hyperoperations defined on a set H are non-degenerate, i.e. $xy\neq\emptyset$ for all $x,y\in H$.

One of the main motivating examples, which also leads to a large class, to define hyperstructures was, has been since the beginning of this theory, the following:

A1.1 EXAMPLE

Let H be a subgroup of a group $(G,\cdot)$. The set of all right cosets $G/H = \{Hg: g\in G\}$ with the hyperoperation

$$(Hg_1)\cdot(Hg_2) = \{ Hg_1hg_2 : h\in H \}$$

becomes a hypergroup.

Every hypergroup which is isomorphic to one given in the above example, is called **D-hypergroup**. The class of D-hypergroups is very large but all hypergroups are not D-hypergroups.

A1.2 DEFINITIONS

A hypergroup $(H,\cdot)$ is called **hypergroup of type U**, if it has a scalar identity e, $x\cdot e=x=e\cdot x$ for all $x\in H$, and the following axiom is valid for all $x,y\in H$,

$$\text{if} \quad x \in x\cdot y, \quad \text{then} \quad y=e.$$

$(H,\cdot)$ is called **hypergroup of type C**, if it has a scalar identity e, and the following axiom is valid for all $x,y,z\in H$

$$\text{if} \quad x\cdot y\cap x\cdot z \neq \emptyset, \quad \text{then} \quad e\cdot y = e\cdot z.$$

A hypergoup of type C is called **cogroup** if, in addition, the next axiom holds

$$|x\cdot y| = |x\cdot z| \quad \text{for all} \quad x,y,z\in H.$$

It is proved that not every cogroup is a D-hypergroup.

A1.3 DEFINITIONS

A hypergroup $(H,\cdot)$ is called **canonical** if it is commutative with a scalar identity, every element x has a unique inverse x' and it is reversible, i.e $z\in xy$ implies $y\in zx'$.

Now let us consider a commutative and associative hyperoperation $(\cdot)$ defined in a set H. For given $a,b\in H$ we set $a/b = \{ c\in H : a\in bx \}$.

The hyperstructure $(H,\cdot)$ is called **join space** if the following additional properties are valid for all $a,b,c,d\in H$

$$a/b \neq \emptyset, \quad \text{and}$$

$$\text{if} \quad a/b \cap c/d \neq \emptyset \quad \text{then} \quad ad\cap bc \neq \emptyset.$$

A join space is called **geometric space** or **join geometry** if, moreover,

$$aa=a \quad \text{and} \quad a/a=a \quad \text{for all} \quad a\in H.$$

The join space is a weaker hyperstructure than the canonical hypergroup. Actually the following is valid:

A join space is a canonical hypergroup if and only if it has a scalar identity.

A multivalued system $(G,\cdot,^{-1},e)$ where

$$e\in G, \quad ^{-1}: G \longrightarrow G, \quad \cdot: G^2 \longrightarrow \mathcal{P}(G)-\{\emptyset\},$$

is called **polygroup** if the following axioms hold for all $x,y,z\in G$:

(i) $(xy)z = x(yz)$

(ii) $xe = x = ex$

(iii) $x\in yz$ implies $y\in xz^{-1}$ and $z\in y^{-1}x$

In other words G is a completely regular, reversible-in-itself multigroup in the sense of Dresher and Ore i.e. the (iii) above axiom holds. The polygroups are also called quasi-canonical hypergroups. The polygroup is a generalization of the canonical hypergroup as well. In fact: a commutative polygroup is a canonical hypergroup.

A hypergroup H is called **feebly canonical** if it is commutative, regular, reversible and satisfies the following condition:

for all $x\in H$, if x',x'' are inverses of x, then

for all $a\in H$ we have $ax' = ax''$.

It is proved that the feebly canonical hypergroups are join spaces.

A subhypergroup U of a hypergroup $(H,\cdot)$ is called left closed with respect to H if for any $a,b\in U$ all solutions of $x\ni ab$ lie in U. U is right closed if all solutions of $y\ni ab$ lie in U. If U is left and right closed then it is called **closed**.

An equivalent definition for closed subhypergroups is the following:

U is closed iff $(H-U)U = U(H-U) = H-U$.

The subhypergroup U of H is called left **ultra-closed** if

$$xU\cap x(H-U) = \emptyset \quad \text{for all} \quad x\in H.$$

The right ultra-closed and the ultra-closed are defined analogously.

Now, we give some definitions of hyperstructures with more than one hyperoperations.

A1.4 DEFINITIONS

A hyperstructure $(R,+,\cdot)$, where $(+)$ is a hyperoperation and $(\cdot)$ is an operation, is called **hyperring in the sense of Krasner** if the following axioms hold:

(i) $(R,+)$ is a canonical hypergroup where the neutral element is denoted by 0.

(ii) $(R,\cdot)$ is a semigroup where 0 is a two-sided absorbing.

(iii) $(\cdot)$ is two-sided (strong)distributive with respect to $(+)$.

A hyperring $(F,+,\cdot)$ is called **hyperfield** in the sense of Krasner if $(F-\{0\},\cdot)$ is a group.

The particular case of this structure, introduced by Krasner, were the residual hyperfields of a valued field.

Let R be a hyperring, then $(M,+,\circ)$ is called **hypermodule** if the following axioms hold:

(i) $(M,+)$ is a canonical hypergroup.

(ii) $\circ: R\times M \longrightarrow M$: $(a,x) \longmapsto M$, is a single valued map such that for all $a,b\in R$ and $x,y\in M$:

$$a\circ(x+y) = a\circ x+a\circ y, \quad (a+b)\circ x = a\circ x+b\circ y$$

$$(a\cdot b)\circ x = a\circ(b\circ x), \quad 0\circ x = 0.$$

A hypermodule M over R is called **hypervector space** if, in addition, R is a hyperfield and the unit 1 satisfies the axiom

$$1\circ x=x \text{ for all } x\in M.$$

We conclude the definitions of the Krasnerian type hyperstructures with a definition of an ordered hyperstructure.

A hyperstrucure $(H,\circ,\vee,\wedge)$ is called lattice hypergroupoid if it satisfies the following axioms:

1. $(H,\circ)$ is a hypergroupoid.
2. $(H,\vee,\wedge)$ is a lattice with regard to the order relation $\leq$.
3. The hyperproduct $x\circ y$ is a closed interval for all $x,y\in H$.
4. For all $x,y,a\in H$ we have

$$(x\vee y)\circ a \supset (x\circ a)\vee(y\circ a) \quad \text{and} \quad a\circ(x\vee y) \supset (a\circ x)\vee(a\circ y).$$

Apart from the definitions of classical hyperstructures based on the canonical hypersructure, the Krasnerian type, several other hyperstructures were introduced and studied. For example, the hypergroup in the sense of Wall, the multiplicative and the additive hyperrings even more general both operations to be hyperoperations, hypervector spaces with the external hyperoperation and so on.

A large number of theorems, profound and important ones on hypergroups, were obtained by Corsini's school. The main tool in this study is the β^* relation or some generalizations of it. This study leads to a number of new classes of hyperstructures and the study of these classes is far to be complete.

Τα ερπετά στήσαν βασίλειο
σε τούτη δω την κορυφή
και απαιτούνε υποκλίσεις
σε τιμιότη ερπετή.

Reptiles built a kingdom
on this peak
and they demand that we bow
before their reptilian honesty.

APPENDIX 2

VERY THIN HYPERSRUCTURES

A2.1 VERY THIN H_v-GROUPS

In the following we deal with a special class of H_v-structures which is nearest, in some sence, to the coresponding stucture, introduced in [40]. According to Definition 1.5.11, a H_v-structure is called **very thin** if and only if all its hyperoperations are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. More precisely, given a H_v-structure defined in a set H, there exists a hyperoperation $(\cdot)$ and a special pair $(a,b)\in H^2$ such that $ab=A$, with $cardA>1$, and all the other products, with respect to all the other hyperoperations, are singletons.

A2.1.1 THEOREM

Let $(H,\cdot)$ be a finite very thin H_v-group, $cardH=n>1$, and let a, b be the only elements of H such that $ab=A$ with $cardA>1$. Then we have the following two cases:

(i) either for all u in H-{a}, ua=a; in which case:

if n=2, then there exists a group (H,*) such that $a*b \in A$ and $x*y = x \cdot y$ for all $(x,y) \in H-\{(a,b)\}$,

if n>2, then a=b, H-{a} is a group and A=H or A=H-{a}.

(ii) or if there exists an element $u \in H$ such that $u \neq a$ and $ua \neq a$, then there exists an almost-associative group (H,*) (i.e. the associativity is valid for all triplets except the ones which contain the product a*b), such that

$a*b \in A$ and $x*y=xy$ for all (x,y) in $H-\{(a,b)\}$.

Proof

First we remark that every element $s \neq a$ is left simplifible, i.e. $sx=sy$ implies $x=y$. This is true since the reproduction axiom is valid and H is finite. Similarly, every $s \neq b$ is right simplifible.

(i) Let ua=a for all $u \in H-\{a\}$. Then if n=2 we have the following two cases, up to isomorphism,

	a	u
a	a,u	a
u	a	u

or

	a	b
a	b	a,b
b	a	b

and the result is obvious since in the first case a*a=u and in the second case a*b=a.

Now, if n>2 then we have ua=a=au, aa = A = H-{a} or H, so a=b. It is easy to verify that H-{a} is a group.

(ii) Let us suppose that there exists $u \in H$ such that $u \neq a$ and $ua \neq a$. Then if n=2, we have the following two cases, up to isomorphism,

	a	b
a	a	a,b
b	b	a

$a \neq b$, u=b and a*b=b

	a	u
a	a,u	u
u	u	a

a=b, u≠b and a*a=a.

If n>2, take x,y∈H such that x≠b≠y and suppose that ax=ay. Then the sets u(ax), u(ay), (ua)x and (ua)y are singletons, so it is obtained that u(ax) = u(ay) = (ua)x = (ua)y. Thus, from the simplifibility x=y.

Let V = H-(H-{a})b. If v belongs to a(H-{b}), then there exists x≠b such that v=ax so, for all y∈H-{x,b}, there exists z∈H-{a} such that zy=v. Therefore, for given y, there exists a unique z and, from simpifibility, it is obtained that the map

$$H-\{x,b\} \longrightarrow H-\{a\} : y \longmapsto z$$

is injective. Therefore, card(H-{a})≠1, because cardH>2. Thus there exists s≠a such that sy≠v for all y∈H-{x,b}. Then we have either sx=v=ax and x≠b which is a contradiction, or sb=v which contradicts to V=H-(H-{a})b. Consequently, V is the result which remais to put a*b=V.

The almost-associativity is obvious. ∎

A2.1.2 THEOREM

Let (H,·) be a finite very thin hypergroup in the sense of Marty. Then there exists an element x of H such that

(i) (H-{a},·) is a group,

(ii) ah=ha=a for every h≠a, and aa = H-{a} or aa=H.

Proof

Let ab=A be the the only one hyperproduct with cardA>1. Let us suppose that a≠b and ba={c}; then a(ba)=ac and (ab)a=Aa. But from the reproduction axiom for the finite hypergroup H, and since a is a right scalar, we have cardAa = cardA > 1. So, from associativity we must have c=b, i.e. ba=b. Similarly, from b(ab)=bA and (ba)b=cb, and since b is a left scalar, we obtain that ba=a. Thus b=a which is a contradiction.

So there exists a unique element a∈H such that card(aa) > 1.

Now, for this special element a and for every $h \neq a$, we have $(aa)h=a(ah)$, where $card(aa)h > 1$, because h is scalar. So $ah=a$.

Similarly, from $h(aa)=(ha)a$ we obtain $ha=a$. Therefore, $ah=ha=a$ for every h of $H-\{a\}$. Finally, from the reproduction axiom for a, and since $x(H-\{a\}) = (H-\{a\})a = a$, we must have $aa = H-\{a\}$ or $aa = H$. That proves (ii).

The proof of (i) is now immediate since every element of $H-\{a\}$ is scalar and $ha=a$ for every $h \neq a$, so we must have

$$h(H-\{a\}) = (H-\{a\})h = H-\{a\}. \qquad \blacksquare$$

If $aa=H-\{a\}$ then H is called very thin hypergroup of the **first kind**, if $aa=H$ then H is called of **second kind**. Remark that the fundamental groups of the first and second kind are respectively $\{\{a\}, H-\{a\}\}$ and $\{H\}$.

Remark. Since $aa=H-\{a\}$ or $aa=H$, it is obtained that all proper subhypergroups of a very thin hypergroup, are groups.

A2.1.3 REMARK

The Theorems A2.1.1 and A2.1.2 give a complet description of the finite very thin H_v-groups and hypergroups. C. Gutan proves these theorems for the infinite case in her paper: Les hypergroupes et les H_v-groupes tres fins, to appear in Rivista Mat. Pura Appl.

A2.1.4 PROPOSITION

Let H be a very thin hypergroup with $|aa|>1$.

(i) Every subhypergroup of H is closed in H.

(ii) There exists only one subhypergroup of H which is ultra-closed in H, this is valid only if H is of first kind and this is the group $H-\{a\}$.

Proof

(i) Let G be a sub(hyper)group of $H-\{x\} \neq G$. Then we have for $g \in G$, $xg=\{x\} \subset H-G$ and, for every $y \in H-G$ with $y \neq x$, the singleton yg is a subset of H-G because otherwise if $yg \subset G$ then we must have $y \in G$ which is a contradiction. Therefore $(H-G)G \subset H-G$.

Conversely, if $y \in H-G$ then from the reproduction axiom $y \in Hg$, i.e. there exists a $h \in H$ such that $y \in hg$. But $h \in H-G$ because, otherwise, if $h \in G$ then $y \in hg \subset G$, a contradiction. Therefore, $(H-G)G = H-G$ and, similarly, $G(H-G) = H-G$. So G is closed in H.

Finally $H-\{x\}$ is also closed in H since

$$x(H-\{x\}) = (H-\{x\})x = x.$$

(ii) Let H be a very thin hypergroup of the first kind. Then $H-\{x\}$ is ultra-closed since we have

$$\left.\begin{array}{l} z(H-\{x\})\cap zx = (H-\{x\})\cap\{x\} = \emptyset \\ (H-\{x\})z\cap xz = (H-\{x\})\cap\{x\} = \emptyset \end{array}\right\} \quad \text{if} \quad z \in H-\{x\}$$

and

$$x(H-\{x\})\cap xx = \{x\}\cap(H-\{x\}) = \emptyset.$$

Now, let G be any subgroup of H with $G\neq H-\{x\}$ if H be of first kind or G be any subgroup oh H, including $H-\{x\}$, if H be of second kind. Then G is not ultra-closed. Indeed, for $G\neq H-\{x\}$ we have $xG=x$ and $x\in x(H-G)$ because there exists a $y\in H-G$ with $y\neq x$, so $x=xy$. Therefore, $xG\cap x(H-G) = \{x\} \neq \emptyset$. Finally, for the group $H-\{x\}$ if H is of second kind, we have again

$$x(H-\{x\})\cap xx = \{x\}\cap H = \{x\} \neq \emptyset. \qquad \blacksquare$$

One could also investigate if an analogous theorem is valid for H_v-groups.

A2.2 VERY THIN H_v-RINGS

A2.2.1 THEOREM

Let $(R,+,\cdot)$ be a ring and $a,b\in R$ be two fixed elements. Consider the hyperoperation $\oplus$ defined by setting

$$a\oplus b=A \quad \text{where} \quad a+b\in A, \ |A|>1 \quad \text{and}$$

$$x\oplus y=x+y \quad \text{for all} \quad (x,y)\neq(a,b).$$

Then $(R,\oplus,\cdot)$ is a very thin additive H_v-ring.

We have same result for the hyperoperation $\odot$ defined by setting

$$a\odot b=A \quad \text{where} \quad a\cdot b\in A, \ |A|>1 \quad \text{and}$$

$$x\odot y=x\cdot y \quad \text{for all} \quad (x,y)\neq(a,b),$$

then $(R,+,\odot)$ is a very thin multiplicative H_v-ring.

Proof
Obvious since the new hyperoperations contain the operations of the ring $(R,+,\cdot)$. $\blacksquare$

A2.2.2 THEOREM

If $(R,+,\cdot)$ is a multiplicative very thin H_v-ring, such that $a\cdot b=A$, $|A|>1$, with $a=0$ or $b=0$, then by setting $a*b=0$ and $x*y=x\cdot y$ for all $(x,y)\neq(a,b)$, the structure $(R,+,*)$ is a ring.

Proof

Let $a=0$, $b\neq 0$. If $x\neq 0$, then $A=0b=(x-x)b \ni xb-xb=0$, so $0\in A$. Moreover, $0x=0$ for all $x\neq b$, because for an element $y\neq 0$, we have $0x=(y-y)x=yx-yx=0$. Similarly, $x0=0$ for all $x\in R$.

Let $x,y,z\in R$. If $0\in\{x,y,z,xy,yz\}$, then from the above properties it is obtained that $x*(y*z) = 0 = (x*y)*z$. Moreover, for the rest cases the associativity is valid since the hyperoperation becomes an operation.

The distributivity is also valid as one can see in an analogous way. Therefore, $(R,+,*)$ is a ring.

The proof for the case $a\neq 0$, $b=0$ is analogous and in the case $a=b=0$, is more simple. ∎

Now, we present some results on hyperrings.

A2.2.3 THEOREM

There exists only one non degenerate multiplicative very thin hyperring, that is the set $R=\{0,a\}$ with hyperoperations defined by the relations

$$0+0 = a+a = 0,\quad 0+a = a+0 = a \quad \text{and} \quad 0\cdot 0 = 0\cdot a = a\cdot 0 = 0,\quad a\cdot a = R.$$

Proof

Let $(R,+,\cdot)$ be a multiplicative very thin hyperring, so $(R,\cdot)$ is a group. Then there are $a,b\in R$ such that $ab=A$, $|A|>1$ and the other products are singletons. Let us suppose $b\neq 0$ and there exists $d\neq 0$ such that $d\neq b$, then $a(d+(b-d))=A$. But $ad+a(b-d)$ is a singleton, so the distributive law is not valid. Therefore with the above assumptions there are no multiplicative very thin hyperrings. Similarly there are no multiplicative very thin hyperrings if $a\neq 0$ and there exists $d\neq 0$ such that $d\neq a$.

From the above it is obtained that $R=\{0,a\}$, $a\neq 0$. ∎

In the case of an additive very thin hyperring $(R,+,\cdot)$, so $(R,+)$ is a very thin hypergroup, from Theorem A2.1.2, it is obtained that there exists an $x\in R$ such that

(i) $(R-\{x\},+)$ is a group

(ii) $x+r=r+x=x$ for every $r\neq x$ and $x+x=R-\{x\}$ or $x+x=R$

Again, in the case $x+x=R-\{x\}$, $(R,+,\cdot)$ is called of **first kind** and in the case $x+x=R$ is called of **second kind**.

In the first kind we have two cases:

Case (a) $xx = r_0 \neq x$.

A2.2.4 THEOREM

For every $r_0 \in R-\{x\}$ such that $r_0+r_0=0$, and only in that case, we have an additive very thin hyperring of the first kind. For those hyperrings we have $xx=r_0$ and the rest products are always zero.

Proof

From the distributive law and especially from $x(x+x) = x(R-\{x\})$ and $xx+xx = r_0+r_0 \neq 0$ we obtain $xr=r_0+r_0$ for every $r \neq x$. But for $r \neq x$ we have $x(x+r) = xx = r_0$ and $xx+xr = r_0+(r_0+r_0)$, so $r_0+r_0=0$. Therefore, $xr=0$ for every $r \neq x$. Moreover, for $r \neq x$ we have $(x+x)r = (R-\{x\})r$ and $xr+xr=0$ so $sr=0$ for every r,s of $R-\{x\}$. This completes the proof. ∎

Case (b) $xx = x$.

From $x(x+x) = x(R-\{x\})$ and $xx+xx = R-\{x\}$ we obtain that for every $r \neq x$ we have $xr \neq x$ and similarly $rx \neq x$. Now let $r,s \in R-\{x\}$, then from $r(x+s)=rx \neq x$ and $rx+rs \neq x$ we obtain $rx=rx+rs$, so $rs=0$. We have two more restrictions: first, from $r(x+x) = r(R-\{x\}) = 0$ and $rx+rx \neq x$ we have $0=rx+rx=(r+r)x$ for all $r \neq x$; second, from $(r+s)x \neq x$ and $rx+sx \neq x$ we have $(r+s)x = rx+sx$ for all $r,s \neq x$.

A2.2.5 THEOREM

Let $R = \{ r \mid r \in R,\ r+r=0 \}$ and $f: R-\{x\} \to R$ be the additive homomorphisms which are also projections. Then we have additive very thin hyperrings of the first kind where $xx=x$ iff we take any of the above f's and define the product as follows:

$rs=0$, $xr=rx=f(r)$, $xx=x$ for every $r,s \in R-\{x\}$.

Proof

It is obtained immediately from the above restrictions and from the associativity, $x(xr) = xf(r) = f(f(r))$ and $(xx)r=xr=f(r)$, f must be a projection. ∎

In the case of the second kind we have again two cases:

Case (a) $xx=r_0*x$.

A2.2.6 THEOREM

The only additive very thin hyperring of the second kind where $xx*x$ is the degenerate one, i.e. all products are equal to zero.

Proof

From $x(x+x) = xR$ and $xx+xx = r_0+r_0$ we obtain that $r_0=0$ and $xR=Rx=0$. From $r(x+x) = rR$ and $rx+rx=0$ we obtain that $rs=0$ for all $r,s\in R$. ∎

Case (b) $xx=x$.

First we have a degenerate case for which $RR=x$.

Now, for an $r\in R$ we have $rx=r'*x$. Then from $r(x+x)=rR$ and $rx+rx=r'+r'$, we have $rs=r'+r'$ for $s\in R$. This means that $r'=0$. Therefore, for any $r\in R$ we have either $rx=x$ or $rR=0$. Moreover, if there exists r such that $rx*x$ then from $(0+r)x=0$ and $0x+rx=0x$ we obtain $0R=0$. Finally if $rx*x$, $sx*x$ then $(r+s)x*x$. We have similar remarks using the right multiplication.

All these remarks give us restrictions on the non-degenerate cases. Here is a construction satisfying the above restrictions:

A2.2.7 THEOREM

Let $(R,+,\cdot)$ be a ring and $f:R \longrightarrow R$ be an endomorphism such that $R/kerf$ be an integral domain. Let $x\notin R$ and consider the very thin hypergroup $(R_x,+)$, where $R_x=R\cup\{x\}$, extending the addition on R_x by setting $r+x=x+r=x$ for all $r\in R$ and $x+x=R_x$. On R_x we define a hyperproduct $(*)$ as follows

$$r*s = f(rs) \quad \text{for all} \quad r,s\in R$$

$$(kerf)*x = x*(kerf) = 0$$

$$(R_x-kerf)*x = x*(R_x-kerf) = x$$

Then $(R_x,+,*)$ is a very thin additive hyperring of the second kind.

Proof

Since $R/kerf$ is an integral domain, we have $r*s = f(r)f(s) \notin kerf$ for every r,s of $R-kerf$, therefore $x*(r*s) = x = (x*r)*s$.

One can easily verify that in all the other cases the associative law for the (*) is also satisfied.

The distributive law is also satisfied but it is not always strong.

The strong left distributivity is not valid in the following cases $x*(r+s)=0$ or x, $x*(x+r)=x$, $r*(x+x)=(f(r)f(R))\cup\{x\}$, $x*(x+x)=\{0,x\}$ for all $r,s \in R\text{-ker}f$. In all those cases the right hand side of distributivity is always equal to R_x. ∎

A2.2.8 OPEN PROBLEMS

Do the above Theorems A2.2.4 - 7 cover all the cases of the very thin hyperrings? If not, find all the rest classes.

Give nessesary and sufficient conditions such that a H_v-ring to be very thin.

Παραμένω στο ρομαντισμό
παραμένω στον ωκεανό
παραμένω στη θύελλα μπροστά
και στα όνειρά μου τα νωπά.

I remain a romantic
I remain in the ocean
I remain in front of the storm
and my fresh dreams.

APPENDIX 3

P-HYPERSTRUCTURES

A3.1 P-H_v-GROUPS

In this appendix we deal with the P-hyperstructures introduced for hypergroups in [30] and generalized in [31]. Several results were published since then. We present some of them here without proofs.

A3.1.1 DEFINITION

Let $(G,\cdot)$ be a semigroup and $P\subset G$, $P\neq\emptyset$. We define the following hyperoperations, called **P-hyperoperations** :

$$\left.\begin{array}{ll} P^* : & xP^*y = xPy \\ P^*_\ell : & xP^*_\ell y = xyP \\ P^*_r : & xP^*_r y = Pxy \end{array}\right\} \quad \text{for all } x,y \in G.$$

Then $\langle G,P^*\rangle$, $\langle G,P^*_r\rangle$, $\langle G,P^*_\ell\rangle$ are called **P-hyperstructures**.

It is immediate that for P^* the associativity is strong, so $\langle G,P^*\rangle$ is a semihypergroup.

If $P\subset Z(G)$, where $Z(G)$ the center of G, then $P^* \equiv P^*_r \equiv P^*_\ell$.

The P-hyperoperations can be defined in groupoids and hypergroupoids as well. However, the study of these artificial hyperstructures is more complicated, so we restrict ourselves on P-hyperstructures defined in semigroups.

The P-hyperoperations can be defined, sometimes, on structures with partial operations. For example the sandwich multiplication on non square matrices, see paragraph 3.5. The same situation appeared in the generalization of the gamma rings in the sense of Barnes.

If $(G,\cdot)$ is a group then it is clear that every P-hyperoperation is of **constant length**, i.e. $|xP^*y| = |P|$ for all $x,y\in G$.

A3.1.2 THEOREM

Let $(G,\cdot)$ be a semigroup, then the P-semihypergroups $\langle G,P^*\rangle$ are H_v-groups (so hypergroups) iff $(G,\cdot)$ is a group.

Proof
If $(G,\cdot)$ is a group then obviously $\langle G,P^*)$ is a hypergroup.
Now, let us suppose that $(G,\cdot)$ is not a group, then there exists an element $x\in G$ such that $xG \subsetneq G$. Remark that the greater P-hyperoperation is for G, i.e. for all $u,v\in G$ and $P\subset G$ it is obtained that $uP^*v \subset uGv$. Therefore it is enough to prove that $\langle G,G^*\rangle$ is not a H_v-quasigroup. From the hypothesis there exists $y\in G-xG$, suppose also that the reproduction axiom is valid for $\langle G,G^*\rangle$. Then

$$y \in xG^*G = xGG \subset xG$$

which is a contradiction. ∎

A subset H of G will be called P-subhypergroup of $\langle G,P^*\rangle$ if $P\subset H$ and $\langle H,P^*\rangle$ is a hypergroup. Similar definition can be given for the other P-H_v-structures. However, one can consider more general P-H_v-structures without assuming $P\subset H$.

A3.1.3 THEOREM

Let $(G,\cdot)$ be a semigroup and $P\cap Z(G) \neq \emptyset$ then

(a) $\langle G,P^*_r\rangle$, $\langle G,P^*_\ell\rangle$ are H_v-semigroups

(b) $\langle G,P^*_r\rangle$ and $\langle G,P^*_\ell\rangle$ are H_v-groups iff $(G,\cdot)$ is a group.

Proof

(a) For all $x,y,z\in G$ we have

$$xP^*_n(yP^*_nz) = xP^*_n(yzP) = xyzPP \quad \text{and}$$

$$(xP^*_ny)P^*_nz = (xyP)P^*_nz = xyPzP.$$

But $P\cap Z(G)\neq\emptyset$, so $zP\cap Pz\neq\emptyset$, therefore

$$xP^*_n(yP^*_nz) \cap (xP^*_ny)P^*_nz \neq \emptyset.$$

(b) The same proof as in Theorem A3.1.2. ∎

A3.1.4 THEOREM

Let $(G_1,\cdot)$, $(G_2,\circ)$ be two groups, $f\in Hom(G_1,G_2)$ and $P\subset G_1$. Then the homomorphism f is a strong homomorphism between the P-hypergroups $\langle G_1,P^*\rangle$ and $\langle G_2,f(P)^*\rangle$. Similar results for the $f(P)^*_n$ and $f(P)^*_\ell$, are valid.

Proof

We have for all x,y in G_1

$$f(xP^*y) = f(xPy) = f(x)\circ f(P)\circ f(y) = f(x)f(P)^*f(y).$$

The same proof for the $f(P)^*_n$ and $f(P)^*_\ell$. ∎

We observe that every $f\in EndG$ of a group $(G,\cdot)$ induces a strong homomorphism on the P-hypergroups $\langle G,P^*\rangle$, $\langle G,f(P)^*\rangle$ for any $P\subset G$. Similarly, every $f\in AutG$ induces an isomorphism on $\langle G,P^*\rangle$, $\langle G,f(P)^*\rangle$. In the special case $f(P)=P$ we have respectively an endomorphism or an automorphism in $\langle G,P^*\rangle$.

As an example of the last case we give the following:

If $P\subset G$ then every element of the centralizer of P, $x\in C_G(P)$, induces an inner automorphism $g\longmapsto x^{-1}gx$ which is an automorphism of $\langle G,P^*\rangle$.

A general useful result on P-hypergroups defined on the same group $(G,\cdot)$, using left and right translations, is the following:

A3.1.5 THEOREM

Let $(G,\cdot)$ be a group and $P\subset G$. Then, for every $g\in G$, it is obtained

$$\langle G,P^*\rangle \cong \langle G,(gP)^*\rangle \cong \langle G,(Pg)^*\rangle.$$

Proof

Consider the right, by g, translation R_g in $(G,\cdot)$, which, obviously, is an 1-1 and onto mapping. Then for all $x,y\in G$ we have

$$R_g(x(gP)^*y) = (xgPy)g = (xg)P(yg) = R_g(x)P^*R_g(y).$$

Hence, R_g establises an isomorphism on $\langle G,(gP)^*\rangle$ and $\langle G,P^*\rangle$.

Similarly, L_g establises an isomorphism on $\langle G,(Pg)^*\rangle$ and $\langle G,P^*\rangle$.

Thus every left or right translation of the set P gives an isomorphic hypergroup. ∎

Remark that if $(G,\cdot)$ is a semigroup then strong homomorphic P-semihypergroups are obtained.

A3.1.6 DEFINITION

Let $(G,\cdot)$ be a group and $P\subset G$. It is called **P_c-hyperoperation** the one defined by the relation

$$xP_c^*y = yPx \quad \text{for all} \quad x,y\in G.$$

The $\langle G,P_c^*\rangle$ becomes a hypergroup called P_c-hypergroup.

Consider the surjection $f: G \longrightarrow G: x \longmapsto x^{-1}$ then we have

$$f(xP^*y) = f(xPy) = y^{-1}P^{-1}x^{-1} = f(x)(P^{-1})_c^*f(y) \quad \text{for all } x,y\in G.$$

Therefore, we have the isomorphism $\langle G,P^*\rangle \cong \langle G,(P^{-1})_c^*\rangle$.

Let $(G,\cdot)$ be a group and $P\subset G$. The element $w\in G$ is a left unit in $\langle G,P^*\rangle$ if $wP^*x\ni x$ for all $x\in G$. But for every $x\in G$ we can find $p_x\in P$ such that $wp_x=e$, where e is the identity of G. Then $w=p_x^{-1}$ and we have, for every $y\in G$,

$$wP^*y = p_x^{-1}P^*y = p_x^{-1}Py \ni y$$

Therefore all left units of $\langle G,P^*\rangle$ are elements of the set P^{-1}.

Conversely, for every $p\in P$ and for every $x\in G$ we have $p^{-1}P^*x=p^{-1}Px\ni x$.

So we deduce that the set of all units, which here are two sided, is exactly the set P^{-1}.

Let $x\in G$ and $p_o\in P$. Then an element $x'\in G$ is a left inverse of x with respect to the unit p_o^{-1} if $x'P^*x\ni p_o^{-1}$. So there must exist a $p_x\in P$ such that $x'p_xx=p_o^{-1}$ and thus $x'\in p_o^{-1}x^{-1}P^{-1}$. Conversely, for every element $p_o^{-1}x^{-1}p_1^{-1}\in p_o^{-1}x^{-1}P^{-1}$ we have

$$(p_o^{-1}x^{-1}p_1^{-1})P^*x = p_o^{-1}x^{-1}p_1^{-1}Px \ni p_o^{-1}x^{-1}x = p_o^{-1}.$$

Therefore the left inverse elements of x with respect to the unit p_o^{-1} are exactly the elements of the set $p_o^{-1}x^{-1}P^{-1}$. Similarly, the set of right inverse elements of x with respect to the unit p_o^{-1} is $P^{-1}x^{-1}p_o^{-1}$. Moreover, the set of inverse elements of x with respect to some unit is $P^{-1}x^{-1}P^{-1}$. Notice that the element $p_o^{-1}x^{-1}p_o^{-1}$ is always a two sided inverse to x with respect to p_o^{-1}.

Recall that a hypergroup $\langle H,\cdot\rangle$ with unit elements is called reversible in itself when any relation $m_1\cdot m_2\ni m_3$ implies that there exists a left inverse m_1^{-1} and a right inverse m_2^{-1} such that

$$m_2\in m_1^{-1}\cdot m_3, \quad m_1\in m_3\cdot m_2^{-1}.$$

A3.1.7 THEOREM

Let G be a group and $P\subset G$. Then the P-hypergroup $\langle G,P^*\rangle$ is reversible in itself.

Proof

Let us suppose that the following relation is valid for $m_1,m_2,m_3\in G$

$$m_1P^*m_2 \ni m_3 \quad \text{or} \quad m_1Pm_2 \ni m_3$$

then there exists an element $p\in P$ such that $m_3=m_1pm_2$. Let m_1^{-1},m_2^{-1} be the inverse elements of m_1,m_2 in the group $(G,\cdot)$. We can take the following two-sided inverses of m_1,m_2 respectively, with respect to the unit element p^{-1} of the P-hypergroup $\langle G,P^*\rangle$:

$$\underline{m_1^{-1}} = p^{-1}m_1^{-1}p^{-1}, \quad \underline{m_2^{-1}} = p^{-1}m_2^{-1}p^{-1}.$$

Then $\underline{m_1^{-1}}P^*m_3 = p^{-1}m_1^{-1}p^{-1}Pm_3 \ni p^{-1}m_1^{-1}m_3 = m_2$.

Similarly, $m_3P^*\underline{m_2^{-1}} \ni m_1$. ∎

A3.1.8 THEOREM

Let G be a group and PcG. If H is a subgroup of G with PcH, then the P-subhypergroup $\langle H,P^*\rangle$ of the P-hypergroup $\langle G,P^*\rangle$ is ultra-closed.

Proof

For any $x\in G$ we obviously have

$$xP^*H\cap xP^*(G-H) = \varnothing = HP^*x\cap(G-H)P^*x \ . \qquad \blacksquare$$

If H_1, H_2 are subhypergroups, we denote $[H_1,H_2]$ the set of all elements which belong in some product of elements from H_1,H_2. For P-hypergroups the sets $H_1\cap H_2$ and $[H_1,H_2]$ are subhypergroups. So from the Theorem 8 in [8] we deduce that the P-hypergroup G has unique co-set expansions with respect to H,

$$G = (HP^*x_1)\oplus(HP^*x_2)\oplus \ \ldots \ , \qquad G = (y_1P^*H)\oplus(y_2P^*H)\oplus \ \ldots$$

Finally, there is a unique double co-set expansion

$$G = (HP^*g_1P^*H)\oplus(HP^*g_2P^*H)\oplus \ . \ . \ .$$

In [7] some important results for P-hypergroups were obtained from abelian groups $(G,\cdot)$ using the P^* hyperoperation. Up to the end of this paragraph we will present these results.

Denote G_A the P-hypergroup with respect to the subset A={a,b}, i.e. the hyperoperation is of constant length 2. Let's denote by $O(ab^{-1})$ the order of the element ab^{-1} in $(G,\cdot)$. Then we have the following:

A3.1.9 THEOREM

If G is a finite commutative group and A={a,b}, then

$$|G_A/\beta^*| = |G|/O(ab^{-1})$$

Using Theorem A3.1.4 and some lemmas, the following theorem is obtained, which gives the number of P-hypergroups of length two defined on finite cyclic groups.

A3.1.10 THEOREM

If G is a cyclic group with $|G|=n$, then there are, up to isomorphism, as many P-hypergroups of length two as divisors of n, except 1. In the special case where n is prime, there exists only one P-hypergroup of length two.

Using the above theorem and studying the rest of the cases, one obtains the number of P-hypergroups of length two, constructed from abelian groups of order less than 12 :

A3.1.11 THEOREM

The number of P-hypergroups of length two constructed of abelian groups of order n, up to 11, is given in the following table

order	2	3	4	5	6	7	8	9	10	11
number	1	1	2	1	3	1	5	4	3	1

A3.2 P-H_v-RINGS

In the following we consider a ring $(R,+,\cdot)$ and we take $P_1,P_2 \subset R$ with $P_1 \neq \emptyset \neq P_2$. Using the addition, only one P-hyperoperation is defined, the P_1^*. Using the multiplication we can define the three hyperoperations P_2^*, P_{2n}^* and $P_{2\ell}^*$. We will study only the hyperoperation $xP_{2n}^*y = xyP_2$ for all x,y in R, since the results for the $P_{2\ell}^*$ are similar and for P_2^* are much simpler.

A3.2.1 THEOREM

The hyperoperation P_{2n}^* is weak distributive to the P_1^* iff $0 \in P_1$.

Proof

Suppose $0 \in P_1$. Then for all x,y,z in R we have

$$xP_{2n}^*(yP_1^*z) = x(y+z+P_1)P_2$$

and

$$(xP_{2n}^*y)P_1^*(xP_{2n}^*z) = xyP_2+xzP_2+P_1$$

We obtain

$$x(y+z)P_2 \subset (x(y+z+P_1)P_2) \cap (xyP_2+xzP_2+P_1).$$

and we have the same result for the right distributivity.

Now let us suppose that the distributivity is valid. Then,

$$0P^*_{2\cap}(0P^*_1 0) = 0P^*_{2\cap}P_1 = \{0\}$$

and

$$(0P^*_{2\cap}0)P^*_1(0P^*_{2\cap}0) = 0P^*_1 0 = P_1$$

So, $0 \in P_1$. ∎

A3.2.2 THEOREM

Let $(R,+,\cdot)$ be a ring and $P_1, P_2 \subset R$, with $P_1 \neq \emptyset \neq P_2$. If $0 \in P_1$ and $P_2 \cap Z(R) \neq \emptyset$ then the $\langle R, P^*_1, P^*_{2\cap}\rangle$ is a H_v-ring.

Proof

Immediate from Theorems A3.1.3 and A3.2.1 ∎

A3.2.3 THEOREM

Let $\langle R, P^*_1, P^*_2\rangle$ be a H_v-ring with P-hyperoperations P^*_1, P^*_2 over the ring $(R,+,\cdot)$. Consider the subgroup $\langle P_1\rangle$ of $(R,+)$ generated by P_1, then for all a in R

$$\beta^*_+(a) = \langle P_1\rangle + a$$

and R/β^*_+ is a multiplicative H_v-ring with the inclusion distributivity.

Proof

We denote by $\beta^*_+(a)$ the fundamental class of $a \in R$ and by $\sum^*$ any hypersum with respect to the P^*_1.

Let $a \in R$ and $x \in \beta^*_+(a)$; then there exist $z_1, \ldots, z_{n+1} \in R$, with $z_1 = x$, $z_{n+1} = a$ and there are $y_j \in R$ and finite sets of indices I_i, $i=1,\ldots,n$ such that

$$\{z_i, z_{i+1}\} \subset \sum^*_{j\in I_i} y_j \quad \text{for} \quad i=1,\ldots,n$$

Set $u_i = \sum_{j\in I_i} y_j$ and $s_i = \operatorname{card} I_i$. Then,

$$\{z_i, z_{i+1}\} \subset u_i + (s_i - 1)P_1 \quad \text{for} \quad i=1,\ldots,n$$

Therefore, for $i = 1,\dots,n-1$ we have

$$z_{i+1} \in (u_i+(s_i-1)P_1) \cap (u_{i+1}+(s_{i+1}-1)P_1)$$

and so, $u_i \in z_{i+1}-(s_i-1)P_1$; consequently,

$$u_i \in u_{i+1}+(s_{i+1}-1)P_1-(s_i-1)P_1 .$$

From this relation we obtain

$$u_1 \in u_n+(s_2+s_3+\cdots+s_n-n+1)P_1-(s_1+s_2+\cdots+s_{n-1}-n+1)P_1$$

But $z_{n+1} \in u_n+(s_n-1)P_1$, so $u_n \in z_{n+1}-(s_n-1)P_1$.

Moreover, $z_1 \in u_1+(s_1-1)P_1$. Thus we have

$$z_1 \in z_{n+1}+(s_1+s_2+\cdots+s_n-n)P_1-(s_1+s_2+\cdots+s_n-n)P_1$$

Finally,

$$x \in a+(s_1+\cdots+s_n-n)(P_1-P_1)$$

That means that $x \in \langle P_1\rangle+a$, so $\beta^*_+(a) \subset \langle P_1\rangle+a$.

Now let $a \in R$ and take $x \in \langle P_1\rangle+a$. Then there exists $s \in \mathbb{N}$ such that $x \in s(P_1-P_1)+a$. So,

$$\{x,a\} \subset s(P_1-P_1)+a = aP^*_1(-P_1)P^*_1\cdots P^*_1(-P_1)$$

which means that $x\beta^*_+a$. Thus we proved that $\beta^*_+(a) = \langle P_1\rangle+a$.

The sum $\oplus$ and the product $\odot$ of the elements of R/β^*_+ are defined in the usual manner as in Theorem 1.2.7. So, $\langle R/\beta^*_+,\oplus\rangle$ is a group. Moreover the weak associativity for the $\odot$ is valid because it is valid for all equivalence relations.

Finally, for all x,y,z in R, we have

$$\beta^*_+(x) \odot (\beta^*_+(y) \oplus \beta^*_+(z)) = \beta^*_+(x) \odot \beta^*_+(y+z) =$$

$$= \{\beta^*_+(u) : u \in (\langle P_1\rangle+x)(\langle P_1\rangle+y+z)P_2\}$$

On the other side

$$(\beta^*_+(x) \circ \beta^*_+(y)) \oplus (\beta^*_+(x) \circ \beta^*_+(z)) =$$

$$= \{\beta^*_+(v):\ v\in(<P_1>+x)(<P_1>+y)P_2\} \oplus \{\beta^*_+(w):\ w\in(<P_1>+x)(<P_1>+z)P_2\} =$$

$$= \{\beta^*_+(v+w) :\ v+w \in (<P_1>+x)(<P_1>+y)P_2+(<P_1>+x)(<P_1>+z)P_2\}$$

Consequently, the inclusion distributivity is valid. ■

A3.2.4 THEOREM

Let $(R,+,\cdot)$ be a ring and $P_1,P_2\subset R$ with $0\in P_1\cap P_2$. Consider the H_v-ring $<R,P_1^*,P_2^*>$, then the quotients

$$R/\beta^*_. \qquad \text{and} \qquad (R/\beta^*_+)/\beta^*_\odot$$

are additive H_v-rings with degenerate multiplication.

Proof

We observe that, since $0\in P_2$, for all x,y in R, we have for the first quotient

$$\beta^*_.(x) \circ \beta^*_.(y) = \{\beta^*_.(z) :\ z \in \beta^*_.(x)\beta^*_.(y)P_2\} = \{\beta^*_.(0)\}$$

For the second quotient we have

$$\beta^*_+(x) \circ \beta^*_+(y) = \{\beta^*_+(z) :\ z \in \beta^*_+(x)\beta^*_+(y)P_2\} \ni \beta^*_+(0)$$

So, for all elements of the set

$$(R/\beta^*_+)\circ(R/\beta^*_+),$$

there exists only one fundamental class with respect to $\circ$.

Therefore, for all $x,y\in R$,

$$\beta^*_\odot(\beta^*_+(x))\square\beta^*_\odot(\beta^*_+(y)) = \{\beta^*_\odot(\beta^*_+(0))\} = \{\beta^*_\odot(<P_1>)\}\ .$$ ■

A3.3 CYCLICITY IN P-HYPERSTRUCTURES

A3.3.1 THEOREM

Let $(G,\cdot)$ be a group and $P\subset Z(G)$. If $\langle G,P^*\rangle$ is cyclic then G is abelian.

Proof

If x is a generator of $\langle G,P^*\rangle$ then $Q=P\cup\{x\}$ is a set of generators of $(G,\cdot)$. But P is a subset of the center Z(G), therefore all elements of Q commute pairwise. We conclude: G is abelian. ∎

A3.3.2 THEOREM

Let G be a group and $G\triangleright P$.

(a) If $\langle G,P^*\rangle$ is cyclic then G/P is cyclic.

(b) If G/P is cyclic, with $|G/P|=n$, with generator xP for which $P-\{e\}\subset\{x,x^2,\dots\}P$, then $\langle G,P^*\rangle$ is cyclic.

Proof

(a) It is obvious since x is a generator of $\langle G,P^*\rangle$ then xP is a generator of G/P.

(b) We have $G/P = \{ (xP)^k \mid k=1,\dots,n \} = \{ x^kP \mid k=1,\dots,n \}$ so

$G = (xP)\cup(x^2P)\cup \dots \cup(x^nP) = (xP)\cup x^{[2]}\cup \dots \cup x^{[n]}$.

But $P-\{e\} \subset \{x,x^2,\dots\}P$ so $x(P-\{e\}) \subset \{x^2,x^3,\dots\}P$.

Consequently, $G = \{x\}\cup x^{[2]}\cup \dots \cup x^{[n]}$ and therefore, x is a generator of $\langle G,P^*\rangle$. ∎

A3.3.3 THEOREM

Let $(G,\cdot)$ be a group and $G\triangleright P$ with $P\neq G$. Then the P-hypergroup $\langle G,P^*\rangle$ is not a single-power cyclic.

Proof

Since $P\neq G$, we have $|G/P|>1$ so for every element x and for any power k the set $x^{(k)}=x^kP$ is only one class in G/P and therefore, it can not be equal to G. ∎

From the above theorems we see that we have for some interesting properties on the powers and cyclicity for P-H_v-groups . Especially when the used structure $(G,\cdot)$ is finite, $|G|=n$. We can see, for example, Theorem 2.5.3 which states that for every $x\in G$, the set [x] is a P-subhypergroup. Moreover, remark that the commutativity in G

reduces the three P-hyperoperations into one, as in the case where $(G,\cdot)$ is cyclic.

In this paragraph we present some of the results of the above case. Therefore, in the following we assume that G is cyclic group, so it is cyclic the P-hypergroup, with generator the element $a\in G$; then $G = \{ a, a^2, \ldots, a^n=e \}$. Finally, we denote $a^{r[s]}$ the s power, with respect to P^*, of the element a^r which is the r power of a, with respect to the operation $(\cdot)$.

A3.3.4 LEMMA

Let $(G,\cdot)$, $|G|=n$, be a cyclic group with generator a and $P=\{e,a^r\}$. Then the element a^s of the P-cyclic hypergroup $\langle G,P^*\rangle$ is a generator iff $(r,s,n)=1$.

Proof

The u-th power of the element a^s with respect to P^* is

$$a^{s[u]} = \{ a^{su}, a^{su+r}, \ldots, a^{su+(u-1)r} \} .$$

Thus the elements of the powers of a^s have the form

$$a^{su+vr} \quad \text{where} \quad u\in N_o \quad \text{and} \quad v\in\{0,1,\ldots,u-1\}$$

but we have

$$su+vr \equiv 1 \bmod n \quad \text{iff} \quad \exists t\in Z : su+vr-tn = 1 \quad \text{iff} \quad (r,s,n)=1.$$

So, we can choose appropriate $u,v,t \bmod n$ in order that the relation $a^{su+vr}=a^1=a$ would be valid iff $(r,s,n)=1$. Therefore, the element $a\in G$ belongs to some power of a^s iff $(r,s,n)=1$.

Now, if the element a belongs to some power of a^s, then for all $k\in N_o$, the element $a^k\in G$ belongs to some power of a^s because

$$a^{s(ku)+(kv)r} = a^k.$$

This yields that a^s is a generator of $\langle G,P^*\rangle$ iff $(r,s,n)=1$. ∎

A3.3.5 THEOREM

Let $(G,\cdot)$, $|G|=n$, be a cyclic group with generator a and $P=\{a^u,a^v\}$, $u<v$. Then in the P-cyclic hypergroup of length two $\langle G,P^*\rangle$ the element a^s is a generator iff $(v-u,s,n)=1$.

Proof
It is obtained from the above lemma and from Theorem A3.1.5 using the left translation by $g=a^u$. That is

$$L_{a^u}(\{e,a^{v-u}\}) = \{a^u,a^v\} . \qquad \blacksquare$$

Using the same procedure (Lemma A3.3.4 and Theorem A3.1.5) some other results can be proved. Thus one proves some lemmas for $P=\{e,a^k\}$ and then by using left translations, Theorem A3.1.5, the results are transferred in the general case where $P=\{a^u,a^v\}$.

A3.3.6 THEOREM

Let $P=\{a^u,a^v\}$, $u<v<n$ and denote $[n/2]$ the integer part of $n/2$. Then

(a) The element a^{v-u} is a generator with period $[n/2]$ iff $(v-u,n)=1$.

(b) The element a^{n-v+u} is a generator with period n iff $(v-u,n)=1$.

Recall that if all generators of a H_v-group have the same period, then it is called cyclic with period. On this cyclicity the following can be proved.

A3.3.7 THEOREM

Let $P=\{a^u,a^v\}$, $u<v<n$, then

(a) If $(v-u,n)=1$, $n>2$, then $\langle G,P^*\rangle$ is not cyclic with period.

(b) If $n=rs$, $r\leq s$, $n>2$, then the P-hypergroups $\langle G,\{a^{um},a^{vm}\}^*\rangle$ such that $(v-u,r)=1$, are cyclic with period $r+s-1$.

(c) If $n=r(r+1)$, $n>2$, then $\langle G,\{a^{ru},a^{rv}\}^*\rangle$ such that $(v-u,r+1)=1$, are cyclic with period $2r$.

However, apart from the above two theorems, there are also some questions on this topic as the following ones:

(i) Which generator has the minimal period?

(ii) Does Theorem A3.3.7 cover all the cases for hypergroups which are cyclic with period?

The following exist:

A3.3.8 CONJECTURE

Suppose $r<s<n$ and $u<v<n$. If the element a^{s-r} is a generator of $\langle G,\{a^u,a^v\}^*\rangle$ of period t, then the element a^{v-u} is a generator of $\langle G,\{a^r,a^s\}^*\rangle$ with the same period t.

This conjecture is true up to n=40 as was obtained in tables using a computer. Here we present one of these tables for n=20:

	20		20				20		20		20		20				20		20
20	11	8	8	8	8	8	8	8	11	11	8	8	8	8	8	8	8	11	20
	8		8		8		8		11		8		8		8		8		11
20	8	8	11	8	8	8	11	8	8	11	8	8	8	11	8	8	20	8	8
	8		8		8		8		8		8		8		8		8		8
	8	8	8	8		8	8	8	8		8	8	8	8		8	8	8	8
	8		8		8		11		8		8		8		8		11		8
20	8	8	11	8	8	11	11	8	8	11	8	8	20	8	8	8	8	8	8
	8		8		8		8		8		8		8		8		8		8
20	11	11	8	8	8	8	8	8	11	11	20	8	8	8	8	8	8	8	8
	11		11				11		11		11		11				11		11
20	8	8	8	8	8	8	8	8	20	11	11	8	8	8	8	8	8	11	11
	8		8		8		8		8		8		8		8		8		8
20	8	8	8	8	8	8	20	8	8	11	8	8	11	11	8	8	11	8	8
	8		11		8		8		8		8		11		8		8		8
	8	8	8	8		8	8	8	8		8	8	8	8		8	8	8	8
	8		8		8		8		8		8		8		8		8		8
20	8	8	20	8	8	11	8	8	8	11	8	8	11	8	8	8	11	8	8
	11		8		8		8		8		11		8		8		8		8
20	20	11	8	8	8	8	8	8	8	11	11	8	8	8	8	8	8	8	11

In these square tables, in the r+1, $r\in\{0,1,\ldots,n-1\}$, column the period of the elements $e,a,\ldots,a^{n-1}$ respectively, of the P-hypergroup $\langle G,\{e,a^r\}^*\rangle$ appears. If an element is not a generator then an empty space appears.

For the single-power cyclicity a complete answer with the following theorem is given.

A3.3.9 THEOREM

Let $P=\{a^u,a^v\}$, $u<v<n$, then $\langle G,P^*\rangle$ is single-power cyclic hypergroup iff $(v-u,n)=1$ and in this case all the elements are generators with single-power period n.

Proof

We prove the theorem in the case $P=\{e,a^r\}$, $a^r\neq e$ and the result is transferred by using Theorem A3.1.5.

In the relation

$$a^{s[u]} = \{ a^{su},a^{su+r},\ldots,a^{su+(u-1)r} \}$$

there are at most u different elements, so in order that $\langle G,P^*\rangle$ would be a P-cyclic hypergroup we must have $u\geq n$.

For $u=n$, we have

$$a^{s[n]} = \{ a^{sn},a^{sn+r},\ldots,a^{sn+(n-1)r} \} = \{ e,a^r,\ldots,a^{(n-1)r} \}$$

while, for every $k\in N$, we obtain

$$a^{s[n+k]} = a^{sk}.a^{s[n]}.$$

Therefore, $\langle G,\{e,a^r\}^*\rangle$, $a^r\neq e$, is a single-power P-cyclic hypergroup if and only if exactly the n-th power of a^s is equal to G.

The n elements of $a^{s[n]}$ are different iff $(r,n)=1$, independently of s, and the period of a^s is n. ∎

We conclude this paragraph with the simplest case for $|P|=3$ which is when $P=\{a^r,e,a^{-r}\}$. The main results obtained in [41] are the following:

(a) The element a^s is a generator iff $(r,s,n)=1$.

(b) $\langle G,P^*\rangle$ is single-power cyclic iff $(r,n)=1$, then every element is a generator of period $[n/2]+1$.

(c) If $(r,n)=1$ then $\langle G,P^*\rangle$ is not cyclic with period.

Using Theorem A3.1.5 one transfers these results to every P-cyclic hypergroup with $|P|=3$ where P has the form $\{a^{t+r},a^t,a^{t-r}\}$. However, these results do not cover so many cases as in length two.

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SYMBOLS

WASS, 2

COW, 2

$\leq$, 4

AutH, 4

β, β^* , 6

γ, γ^* , 10, 11

ω, ω_H , 13

$|\omega_H|$-H_v-group, 13

μ-equivalence, 14

S_H , 18

ℓ_X , 41

L_a ,42

$A \hat{\times} B$, 43

β^*_+ , β^*_- , 51

γ^*_1 , γ^*_2 , 53

φ, φ_+, φ^* , 55

ω, ω_+ , ω^* , 55

A_ℓ, A_n , 61

D_ℓ, D_n , 61

ε^* , 62

P^* , 65

θ-WASS, θ, 78

M_X , $\underset{\rightarrow}{M}_X$, 86

$\bar{f}$, 86

dap , 87

M_θ , 91

* , 94

T^* , 98

f_ε , 102

M_U , 103

Tr, tr_X , tr_b , tr_φ , 133

X_T , X_{Tb} , 134

a/b , 142

P^* , P^*_n , P^*_ℓ , 154

I N D E X

absorbing, 61
arrow, 86
associated, 86
associated H_v-rings, 58, 114
associative, 78
automorphism, 4

basic trace, 133
basic character, 134
b-structure, 21
b-operation, 21

canonical, 28, 142
closed, 143
cogroup, 142
commutative H_v-group, 2
complete, 20, 60
condition θ, 91
constant length, 158
constructions S_1-S_5 , 22 - 25
converse, 86
convolution, 79
core, 13
COW, 2
COW H_v-ring, 2
cyclic, 48
cyclic with period, 48

degree, 110
deleting arrows procedure, 87
destination, 61, 62
D-hypergroup, 142
direct sum, 135

equal, 4
extension, 101

faithful, 98, 110
feebly canonical, 143
first, second kind, 153
form left, right, 5
fundamental character, 134
fundamental group, ring, 6,10
fundamental map, 13, 77
fundamental property, 3
fundamental relations 6,10,13,54
fundamental trace, 133

generalized permutation, 85
generates, 63
generator, 48
geometric space, 143
good condition, 98, 110
greater, 4

H_b-structure, 21
H_b-group, 21
H_b-ring, 21
height, 89
H_v-algebra, 81
H_v-canonical hypergroup, 28
H_v-commutative group, 2
H_v-field, 13
H_v-group, 2, 21
H_v-homomorphism, 4

H_v-ideal, 51
H_v-matrix, 109
H_v-matrix representation, 110
H_v-module, 13, 62
H_v-ring, 2
H_v-ring additive, 51
H_v-ring multiplicative, 51
H_v-semigroup, 2
H_v-semiring, 2
H_v-single group, ring, 20
H_v-skew field, 13
H_v-structure, 2
H_v-quasigroup, 2
H_v-vector space, 13, 62
hyperfield, 144
hypergroupoid H_v-algebra, 81
hypergroup, 1
hypergroup of type C, U, 142
hyperring, 144
hypervector space, 144

incidence, 26
incidence matrix, 126
inclusion homomorphism, 4
inclusion representation, 98, 110
independent, 63
induced, 102
inner, 86
inverse, 132
invertible, 132
irreducible, 132, 135, 136
isomorphism, 4

join space, 142
join geometry, 143

left fundamental hypergroup, 41
less, 4

minimal, 4, 86, 128

n-ary circle hyperoperation, 6

partial identity, 93
parenthesis on the left, right, 5
P_c-hyperoperation, 160
period, 48
polygroup, 39, 143
powers, 48
P-hyperalgebra, 65
P-hyperoperation, 27, 157
P-hyperstructure, 27
proper, 6, 86

reducible, 132, 135, 136
representation, 98
reproductive, 85
reproductivity of classes, 20

semidirect hyperproduct, 43
scalar, 18
S-construction, 26, 58
S-hyperoperation, 26, 58
single, 18
single-power 48
strong homomorphism, 4
sub-H_v-group, 2
subpermutation, 86

total, 26
type a, b, c, 59

ultra-closed, 143
unit H_v-matrix, 131
union, 4
uniting elements, 67
universal, 86

very thin, 26, 147

WASS, 2
weak associativity, 2
weak commutativity, 2
weak distributive, 2
weak homomorphism, 4